7. Übungsblatt zur Vorlesung **SS 2019**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Mittwoch, den 26. Juni 2019 16:00 Uhr

Exercise 1: Gravitational Waves **(12 Points)**

(a) Consider small perturbations $h_{\mu\nu}$ around the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1}$$

If the perturbation is rewritten in the form of

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\rho}_{\rho}, \tag{2}$$

the Einstein tensor is given by

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\gamma \partial_\nu \bar{h}^{\gamma}_{\mu} + \partial^{\gamma}\partial_\mu \bar{h}_{\nu\gamma} - \partial_\gamma \partial^{\gamma} \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\gamma \partial^{\delta} \bar{h}^{\gamma}_{\delta}\right). \tag{3}$$

How do the fields $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$ transform under coordinate transformations given by small shifts $x^\mu \to x^\mu + \epsilon^\mu(x)$? Notice the similarity to gauge transformations of the electromagnetic potential A_μ.

(b) The harmonic gauge fixing condition reads:

$$\partial_\mu h^{\mu}{}_{\nu} = \frac{1}{2}\partial_\nu h^{\rho}_{\rho}. \tag{4}$$

This condition admits additional gauge transformations which further reduce the number of independent polarizations of the gravitational wave to two. What condition is obeyed by the ϵ_μ that corresponds to these residual transformations?

(c) Show that the harmonic gauge fixing condition implies the Lorenz gauge for the field $\bar{h}_{\mu\nu}$, i.e. $\partial^\mu \bar{h}_{\mu\nu} = 0$. Write down the Einstein field equations in vacuum for the field $\bar{h}_{\mu\nu}$ in Lorenz gauge.

A solution of the Einstein equation for $\bar{h}_{\mu\nu}$ in Lorenz gauge is

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left[\int dk A_{\mu\nu} \exp\left(-i k_\rho x^\rho\right)\right]. \tag{5}$$

(d) Due to the Lorenz gauge condition four degrees of freedom of the solution above can be eliminated. Give the condition on the parameters $A_{\mu\nu}$ resulting from the Lorenz gauge condition.
How many degrees of freedom are left?

(e) In Lorenz gauge, the gauge functions ϵ_μ fulfill a wave equation themselves, i.e. $\partial_\nu \partial^\nu \epsilon_\mu = 0$. This equation is solved by

$$\epsilon^\mu = \mathrm{Re}\left[\int dp C^\mu \exp\left(-i p_\rho x^\rho\right)\right], \tag{6}$$

with the free parameters C^μ. Show explicitly that the coefficients C^μ can be chosen in a way that

$$\bar{h}^\rho_\rho = 0 \quad \text{and} \quad \bar{h}^\mu_0 = 0\,, \tag{7}$$

are satisfied. This is called *traceless transverse gauge*. How many degrees of freedom are left?

Exercise 2: Canonical energy momentum tensor **(8 Points)**

The celebrated theorem by Emmy Noether states that every symmetry of the action implies the conservation law for the corresponding current. Recall that the action for the field theory in $3+1$ dimensions is given by $S = \int d^4x \mathscr{L}$, where $\mathscr{L}(\phi, \partial_\mu\phi)$ is the Lagrangian density, hereafter called simply the Lagrangian. In particular, a symmetry transformation that leaves the Lagrangian invariant, $\delta\mathscr{L} = 0$, will leave the action invariant as well.

One finds that the variation of the Lagrangian under the infinitesimal variations of the fields and their first derivatives is:[1]

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathscr{L}}{\partial(\partial\phi(x))}\partial_\mu\delta\phi(x). \tag{8}$$

The equation of motion for the field ϕ is derived from the action principle:

$$\frac{\delta S}{\delta\phi} = 0, \tag{9}$$

where

$$\frac{\delta S}{\delta\phi} = \frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}. \tag{10}$$

Inserting this into Eq.(8) one obtains

$$\delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) + \frac{\delta S}{\delta\phi}\delta\phi. \tag{11}$$

If the equation of motion is satisfied and the infinitesimal symmetry transformation leaves the Lagrangian invariant, $\delta\mathscr{L} = 0$, there is a conserved current:

$$\partial_\mu j^\mu = 0, \text{ where } j^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi. \tag{12}$$

It can happen that the infinitesimal symmetry transformation does not leave Lagrangian invariant but shifts it by a total derivative of a vector, $\delta\mathscr{L} = \partial_\mu K^\mu$. In this case there is a conserved current

$$\tilde{j}^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi - K^\mu. \tag{13}$$

(a) As an example, consider the infinitesimal spacetime translations by a constant vector a_μ, that is $\phi \to \phi(x-a)$. What is the corresponding variation of the field ϕ?

(b) Under this transformation the variation of the Lagrangian is $\delta\mathscr{L} = -a^\mu\partial_\mu\mathscr{L}$. Show this explicitly for the case of the free scalar field whose Lagrangian is

$$\mathscr{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2. \tag{14}$$

[1] In what follows we drop the dependence on the coordinates to simplify the notation

(c) We define the *canonical energy-momentum tensor* $S^{\mu\nu}$ in terms of the conserved current that corresponds to spacetime translations:

$$a_\nu S^{\mu\nu} = \tilde{j}^\mu. \tag{15}$$

Write down the form of $S^{\mu\nu}$ in terms of the Lagrangian.

(d) Derive the canonical energy-momentum tensor for the free electromagnetic field using your knowledge of the Lagrangian $\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. What are the deficiencies of the resulting energy-momentum tensor?