

**Exercise 1: Fundamentals of cosmology**

**(10 Points)**

Let  $T_{\mu\nu}$  be the energy-momentum tensor of a perfect fluid, given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (1)$$

where  $\rho$  is the energy density,  $p$  is the isotropic pressure in the fluid's rest-frame and  $u^\mu$  is its four-velocity. Furthermore, the Friedmann–Lemaître–Robertson–Walker (FLRW) metric is given as

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]. \quad (2)$$

You can find the Christoffel symbols and the Ricci tensor on the next page. The equations that relate the evolution of the scale factor  $a(t)$  to the matter content of the universe are called *Friedmann equations*.

(a) Calculate the Ricci scalar  $R$ .

(b) Derive the Friedmann equations.

The first Friedmann equation can be obtained from the 00-component of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (3)$$

Use the trace of (3) and the first equation and to find the second Friedmann equation.

(c) Consider  $\nabla_\mu T^{\mu 0} = 0$  to derive a continuity equation for  $\rho$  and  $p$ . Is this equation independent of the Friedmann equations?

(d) Suppose that pressure and energy density fulfill the equation of state

$$p = w\rho, \quad (4)$$

where  $w$  is a time-independent constant. Use this equation and the continuity equation from (c) to express the energy density in terms of the scale factor  $a$ .

(e) Determine the time dependence of the scale factor  $a = a(t)$  using the Friedmann equations and  $\rho(a)$  for the case  $k = 0$ .

(f) Consider the energy densities of dust, photons and vacuum energy. Find the constant  $w$  for each case. How do the respective energy densities depend on the scale factor  $a$ ? How does the scale factor evolve in time, if the universe is dominated by dust, photons or vacuum energy?

**Exercise 2: Age of the Universe****(5 Points)**

The fractions  $\Omega_i$  of the different energy forms  $\rho_i$  of the critical energy density  $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$  are given as

$$\Omega_i = \frac{\rho_i}{\rho_{\text{crit}}}, \quad (5)$$

with  $i \in \{M, R, \Lambda\}$ ,  $\rho = \sum_i \rho_i$  and  $M$  corresponding to matter,  $R$  to radiation and  $\Lambda$  to vacuum energy.

- Using the first Friedmann equation determine  $\Omega_k + \sum_i \Omega_i$ , with  $\Omega_k = -\frac{k^2}{H^2 a}$ .
- Rewrite the energy density  $\rho$  in terms of the fractions of energy density measured today  $\Omega_{i,0}$  at  $t = t_0$ .
- Give an expression for the age of the universe  $t_0$  in the form of

$$t_0 = \int_{x_1}^{x_2} f(\Omega_{i,0}, x) dx, \quad (6)$$

with  $x = \frac{a}{a_0}$  where  $a_0 = a(t_0)$ . Once again employ the first Friedmann equation.

- Determine the age of a flat
  - matter-dominated universe.
  - radiation-dominated universe.
  - vacuum energy-dominated universe.
- It is possible to estimate the ages of galaxies at high redshift, at the time far in the past when the light we now observe left these galaxies. The radio galaxy 53W091 at redshift  $z = 1.55$  has an estimated age of 3.5 Gyr. Assuming a flat universe with  $\Omega_R = 0$  derive limits on the parameters  $\Omega_k$  and  $\Omega_M$  from this observation.  
*The occurring inequality has to be solved numerically.*

**Exercise 3: Properties of the Einstein equations****(5 Points)**

With the Einstein tensor  $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$  the field equations of general relativity can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (7)$$

Since both  $G_{\mu\nu}$  and  $T_{\mu\nu}$  are symmetric, (7) represents 10 independent equations. The state of a system at a given time can be described by the values of the metric  $g_{\mu\nu}$  and the time derivative  $\partial_t g_{\mu\nu}$  – analogously to a state consisting of coordinates  $x^i$  and momenta  $p^i$  in classical mechanics. The evolution of such a state is then governed by Einstein's equations (7).

Show that of the 10 independent equations only 6 describe the dynamical evolution of the state, while the 4 equations given by  $G^{0\nu} = 8\pi G T^{0\nu}$  merely serve as constraints on the initial data.

In order to do so, use the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0 \quad (8)$$

to show that the expression  $\partial_t G^{0\nu}$  contains no third-order time derivatives and thus no second-order time derivatives occur in  $G^{0\nu}$ . From this result, conclude that only six equations describe the dynamical evolution of the metric.

Discuss why this situation is analogous to that in electromagnetism.

## Appendix

You have already calculated the non-vanishing Christoffel symbols of the FLRW metric on sheet 3. They are given by

$$\begin{aligned}\Gamma_{rr}^t &= \frac{a\dot{a}}{1-kr^2}, & \Gamma_{\theta\theta}^t &= a\dot{a}r^2, & \Gamma_{\phi\phi}^t &= a\dot{a}r^2 \sin^2\theta, \\ \Gamma_{rr}^r &= \frac{kr}{1-kr^2}, & \Gamma_{\theta\theta}^r &= -r(1-kr^2), & \Gamma_{\phi\phi}^r &= -r(1-kr^2) \sin^2\theta, \\ \Gamma_{r\theta}^\theta &= \Gamma_{r\phi}^\phi = \frac{1}{r}, & \Gamma_{\phi\phi}^\theta &= -\sin\theta \cos\theta, & \Gamma_{\theta\phi}^\phi &= \cot\theta \\ \Gamma_{tr}^r &= \Gamma_{t\theta}^\theta = \Gamma_{t\phi}^\phi = \frac{\dot{a}}{a},\end{aligned}\tag{9}$$

or related to these by symmetry. The nonzero components of the Ricci tensor are

$$\begin{aligned}R_{tt} &= -3\frac{\ddot{a}}{a}, & R_{rr} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}, \\ R_{\theta\theta} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k), & R_{\phi\phi} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \sin^2\theta.\end{aligned}\tag{10}$$