6. Übungsblatt zur Vorlesung **SS 2016**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 28. Juni 2016 16:00 Uhr

Exercise 1: Gravitational Waves **(10 Points)**

(a) Consider small perturbations around the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1}$$

How does the field $h_{\mu\nu}$ transform under coordinate transformations given by small shifts $x^\mu \to x^\mu + \epsilon^\mu(x)$? Notice the similarity to gauge transformations of the electromagnetic potential A_μ.

(b) The harmonic gauge fixing condition reads:

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h. \tag{2}$$

This condition admits additional gauge transformations which further reduce the number of independent polarizations of the gravitational wave to two. What condition is obeyed by the ϵ_μ that corresponds to these residual tranfsormations?

(c) Consider a gravitational wave propagating in the $z-$direction. Its assumed 4-momentum reads $k^\mu = (\omega, 0, 0, \omega)$. The corresponding metric perturbations are specified by: $h_{\mu\nu} = \epsilon_{\mu\nu}\sin(\omega(t-z))$. The transverse-traceless gauge $k^\mu\epsilon_{\mu\nu} = 0$ then implies two non vanishing independent components of the polarisation tensor $\epsilon_{11} = -\epsilon_{22} = \epsilon_+$ and $\epsilon_{12} = \epsilon_{21} = \epsilon_\times$. Before the arrival of the wave, two test masses A and B at the separation $r^\mu = x^\mu_A - x^\mu_B = (0, \Delta x_1, \Delta x_2, 0)$ are at rest. Derive the proper distance between the test masses A and B as a function of time during the propagation of the wave.

Exercise 2: Canonical energy momentum tensor **(10 Points)**

The celebrated theorem, due to Emmy Noether states that every symmetry of the action implies the conservation law for the corresponding current. Recall that the action for the field theory in $3+1$ dimensions is given by $S = \int d^4x\mathscr{L}$, where $\mathscr{L}(\phi, \partial_\mu\phi)$ is the Lagrangian density, hereafter called simply the Lagrangian. In particular, the symmetry transformation that leaves the Lagrangian invariant, $\delta\mathscr{L} = 0$ will obviously leave the action invariant, as well.
One finds that the variation of the Lagrangian under the infinitesimal variations of the fields and their first derivatives is:[1]

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathscr{L}}{\partial(\partial\phi(x))}\partial_\mu\delta\phi(x). \tag{3}$$

The equation of motion for the field ϕ is derived from the action principle:

$$\frac{\delta S}{\delta\phi} = 0, \tag{4}$$

[1] In what follows we drop the dependence on the coordinates to simplify the notation

where

$$\frac{\delta S}{\delta\phi} = \frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}. \tag{5}$$

Inserting this into Eq.(3) one obtains

$$\delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) + \frac{\delta S}{\delta\phi}\delta\phi. \tag{6}$$

If the equation of motion is satisfied and the infinitesimal symmetry transformation leaves the Lagrangian invariant, $\delta\mathscr{L} = 0$, there is a conserved current:

$$\partial_\mu j^\mu = 0, \text{ where } j^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi. \tag{7}$$

It can happen that the infinitesimal symmetry transformation does not leave Lagrangian invariant but shifts it by a total derivative of a vector, $\delta\mathscr{L} = \partial_\mu K^\mu$. In this case there is a conserved current

$$\tilde{j}^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi - K^\mu. \tag{8}$$

(a) As an example, consider the infinitesimal spacetime translations by a constant vector a_μ, that is $\phi \to \phi(x-a)$. What is the corresponding variation of the field ϕ?

(b) Under this transformation the variation of the Lagrangian is $\delta\mathscr{L} = -a^\mu\partial_\mu\mathscr{L}$. Show this explicitly for the case of the free scalar field whose Lagrangian is

$$\mathscr{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2. \tag{9}$$

(c) We define the *canonical energy-momentum tensor* $S^{\mu\nu}$ in terms of the conserved current that corresponds to spacetime translations:

$$a_\nu S^{\mu\nu} = \tilde{j}^\mu. \tag{10}$$

Write down the form of $S^{\mu\nu}$ in terms of the Lagrangian.

(d) Derive the canonical energy-momentum tensor for the free electromagnetic field using your knowledge of the Lagrangian $\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. What are the deficiencies of the resulting energy-momentum tensor?