5. Übungsblatt zur Vorlesung **SS 2019**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 04. Juni 2019 16:00 Uhr

Exercise 1: Properties of the Riemann Tensor **(8 Points)**
At each and every point p there exists a coordinate system $x^{\hat{\alpha}}$ in which the metric takes its canonical form and all first derivatives vanish:

$$g_{\hat{\alpha}\hat{\beta}}(p) = \eta_{\hat{\alpha}\hat{\beta}}, \quad \partial_{\hat{\mu}} g_{\hat{\alpha}\hat{\beta}}(p) = 0, \tag{1}$$

where η denotes the Minkowski metric. Such coordinates are called *locally inertial coordinates*. The second derivatives $\partial_{\hat{\mu}}\partial_{\hat{\nu}} g_{\hat{\alpha}\hat{\beta}}(p)$ do not vanish in general.

(a) Consider the coordinate transformation law for the metric in four dimensions

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu} \tag{2}$$

and expand both sides up to first order in Taylor series in the coordinates $x^{\hat{\mu}}$. Argue by considering the degrees of freedom that one can always choose the coordinates to fulfill Eq. (1).

The curvature is quantified by the *Riemann tensor* (or curvature tensor) with components $R^{\mu}{}_{\nu\rho\lambda}$. These components can be derived from the connection via

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,. \tag{3}$$

(b) Show the following symmetry relations of the curvature tensor $R_{\mu\nu\rho\lambda}$:

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda} \tag{4}$$

$$R^{\mu}{}_{\nu\rho\lambda} = -R^{\mu}{}_{\nu\lambda\rho} \tag{5}$$

$$R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu} \tag{6}$$

$$R^{\mu}{}_{\nu\rho\lambda} + R^{\mu}{}_{\lambda\nu\rho} + R^{\mu}{}_{\rho\lambda\nu} = 0 \tag{7}$$

To show equations (4) and (6) it is useful to employ *Riemann normal coordinates* at the point p. This coordinate system satisfies the following properties at the point p:

$$\Gamma^{\mu}_{\alpha\beta} = 0, \quad \partial_{\nu}\Gamma^{\mu}_{\alpha\beta} \neq 0 \quad \text{and} \quad \partial_{\mu} g^{\alpha\beta} = 0.$$

Riemann normal coordinates provide a realization of the locally inertial coordinates discussed in part (a).

Exercise 2: Curvature of the Torus **(12 Points)**
The torus can be embedded in three dimensional space with the parametrization

$$\vec{r}(\theta,\varphi) = \begin{pmatrix} \cos\theta(a + r\cos\varphi) \\ \sin\theta(a + r\cos\varphi) \\ r\sin\varphi \end{pmatrix}, \tag{8}$$

where a and r are constants.

(a) Show that the induced metric tensor is given by

$$g_{\theta\theta} = (a + r\cos\varphi)^2\,, \quad g_{\varphi\varphi} = r^2\,, \quad g_{\varphi\theta} = 0\,. \tag{9}$$

(b) Determine the geodesic equation starting from

$$\delta \int \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda = 0, \tag{10}$$

and read off the non-vanishing Christoffel symbols.

(c) Give two different forms of geodesics on the torus. Show explicitly that your choice is a solution to the geodesic equations.

(d) Calculate the non-vanishing components of the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{11}$$

Hint: Use the relations from exercise 1 and think about how many independent components there are!

(e) Calculate the Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and the Ricci scalar $R = R^\mu{}_\mu$.

(f) The torus can also be embedded in four dimensional space using the parametrization

$$\vec{r}(u,v) = \begin{pmatrix} A\cos u \\ A\sin u \\ B\cos v \\ B\sin v \end{pmatrix}. \tag{12}$$

Calculate the components of the metric tensor and discuss your results.