4. Übungsblatt zur Vorlesung **SS 2019**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 21. Mai 2019 16:00 Uhr

Exercise 1: Parametrization of geodesics **(4 Points)**

(a) A curve which parallel-transports its own tangent vector fulfills the condition

$$u^\alpha \nabla_\alpha u^\mu = 0\,, \tag{1}$$

where $\vec{u} = d/ds$ is the vector field tangent to the curve, and s parametrizes the curve $x^\mu(s)$. Show that equation (1) implies the geodesic equation.

(b) Show that, if a curve fulfills the equation

$$u^\alpha \nabla_\alpha u^\mu = c(\lambda)\, u^\mu\,, \tag{2}$$

with $\vec{u} = d/d\lambda$, one can always find a parameter $s(\lambda)$ which satisfies $c(s) = 0$. In this case, s is called an *affine parameter*.

(c) In the case of massive particles, extremizing the spacetime interval leads to equation (1) with $\vec{u} = d/d\tau$. Show that, if τ is an affine parameter, $a\tau + b$ is also affine.

Exercise 2: Killing vectors **(8 Points)**

Killing vector fields, or simply Killing vectors, are in one-to-one correspondence with continuous symmetries of the metric on a manifold. Such symmetries are called isometries, and their number is equal to the number of linearly independent Killing vectors. Killing vectors imply the existence of conserved quantities associated with geodesic motion.
Mathematically, isometries are diffeomorphisms that leave the metric unchanged. This means that a diffeomorphism $f : M \to M$ is an isometry if it preserves the metric

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p)\,, \tag{3}$$

where x^μ and y^μ are the coordinates of $p \in M$ and $f(p) \in M$ respectively. Isometries form a group which we study from an infinitesimal point of view.

(a) A vector field $\vec{K} = K^\mu \partial_\mu$ on M is a Killing vector field if the infinitesimal displacement $y^\mu = x^\mu + \epsilon K^\mu$ generates an isometry. Show that this is the case if

$$K^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu} \partial_\mu K^\sigma + g_{\mu\sigma} \partial_\nu K^\sigma = 0\,. \tag{4}$$

These are the *Killing equations*.

(b) Show that the equations

$$\nabla_{(\mu}K_{\nu)} = 0\,, \tag{5}$$

are equivalent to Eq. (4).

(c) Killing vectors can be used to find geodesics on M. Show that the product $\vec{K}\cdot\vec{u}$, where $\vec{u}$ is the vector tangent to a geodesic, is constant along this geodesic.

(d) Show that $\nabla_\mu J^\mu = 0$, where $J_\mu = T_{\mu\nu}K^\nu$ and $T_{\mu\nu}$ is a conserved energy-momentum tensor. Interpret this result.

(e) Consider now the metric of the two sphere $M = S^2$ in local coordinates:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,. \tag{6}$$

What is the maximum number of linearly independent Killing vectors one could in principle find? Derive the Killing vectors for this metric by solving the Killing equations.

Exercise 3: Geodesic of the Schwarzschild Metric **(8 Points)**

Consider the Schwarzschild metric, the solution of the Einstein equations for a spherical mass distribution of mass M. The coordinates can be chosen in such a way that a geodesic always lies in the plane $\theta = \frac{\pi}{2}$. The Schwarzschild metric is given as

$$ds^2 = -\left(1 - \frac{R_S}{r}\right)dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,, \tag{7}$$

with $R_S = 2GM$.
Starting from

$$\delta \int \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda = 0, \tag{8}$$

the geodesic equation can be obtained in the form of

$$0 = \frac{d}{d\tau}\left[g_{\alpha\beta}\frac{dx^\beta}{d\tau}\right] - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,. \tag{9}$$

(a) Determine the geodesic equations for the Schwarzschild metric in case of $\theta = \frac{\pi}{2}$ by use of eq. (9).

(b) Show that the quantities

$$E = \left(1 - \frac{R_S}{r}\right)\frac{dt}{d\tau} \quad \text{and} \quad L = r^2\frac{d\phi}{d\tau} \tag{10}$$

are conserved by the geodesic equations.

For timelike trajectories we can define $d\tau^2 = -ds^2$, while $ds^2 = 0$ for lightlike trajectories.

c) Show that

$$-\kappa = -\left(1 - \frac{R_S}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{R_S}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\,, \tag{11}$$

with $\kappa = 1$ for a timelike trajectory and $\kappa = 0$ for a lightlike trajectory.
In case of a lightlike trajectory think of the parameter τ as any affine parameter parametrizing the trajectory.

d) Using the results of b) and c) verify

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{R_S}{r}\right)\left(\frac{L^2}{r^2} + \kappa\right). \tag{12}$$

Now, consider a radial geodesic, i.e. $d\phi = 0$.

e) Solve the geodesic equation for a lightlike trajectory, i.e. give an expression for $t - t_0$ in terms of $r(t)$ and $r(t_0)$. For which values of τ and t you find $r = R_S$?

f) Determine $\frac{d^2r}{d\tau^2}$ for a timelike trajectory. Which familiar result is obtained?

g) Finally, consider the case of an initially escaping massive particle, i.e. a timelike trajectory with $\frac{dr}{d\tau} > 0$ for $\tau = \tau_0$. For which values of E the particle can actually escape, i.e. reach $r \to \infty$?