3. Übungsblatt zur Vorlesung **SS 2019**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 07. Mai 2019 16:00 Uhr

Exercise 1: Robertson–Walker metric and Christoffel symbols **(8 Points)**

The assumption that the universe is both isotropic and homogeneous led Robertson and Walker to choose the spacetime coordinate system so that the metric takes a simple form. This metric is called the Robertson–Walker metric. Here, we first consider the geometry of a three-dimensional homogeneous and isotropic space. Geometry is encoded in the metric, or equivalently in the line element ds^2. One obvious homogeneous isotropic three dimensional space is flat space, with line element $ds^2 = d\vec{x}^2$. The next possibility is the surface of a 4-sphere with radius a, $ds^2 = d\vec{x}^2 + dy^2$ with $\vec{x}^2 + y^2 = a^2$. It can be shown that the only other possibility is a hyperspherical surface, with line element $ds^2 = d\vec{x}^2 - dy^2$ with $y^2 - \vec{x}^2 = a^2$, where a^2 is a positive constant.

(a) Determine the line element ds^2, which is independent of y, for all three possibilities. Rescale the coordinates with $\vec{x}' = a\vec{x}$ and $y' = ay$. Show that the line element takes the form

$$ds^2 = a^2\left(d\vec{x}^2 + k\frac{(\vec{x}\cdot d\vec{x})^2}{1-k\vec{x}^2}\right), \quad k = \begin{cases} +1 & \text{spherical} \\ 0 & \text{flat} \\ -1 & \text{hyperspherical} \end{cases} . \tag{1}$$

(b) Extend the expression in Eq. (1) to the geometry of spacetime by including this term in the spacetime line element $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$, with a now an arbitrary function of time. Introduce spherical coordinates for the spatial coordinates $\vec{x}$ and determine the components of the metric. Your result should be

$$g_{tt} = -1\,, \quad g_{rr} = \frac{a^2(t)}{1-kr^2}\,, \quad g_{\theta\theta} = a^2(t)r^2\,, \quad g_{\phi\phi} = a^2(t)r^2\sin^2(\theta)\,, \tag{2}$$

and all other components should vanish.

All ways in which curvature manifests itself rely on a *connection*, $\Gamma^\lambda_{\mu\nu}$. There is a unique connection which can be constructed from the metric, and it is encapsulated in the *Christoffel symbols* given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{3}$$

The fundamental use of a connection is to take a covariant derivative ∇_μ (generalization of partial derivative)

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma}V^\sigma\,. \tag{4}$$

One can show that the covariant derivative of a tensor is again a tensor, unlike the partial derivative.

(c) Compute the Christoffel symbols for the Robertson–Walker metric using Eq. (3).

Exercise 2: Geodesics on the earth **(8 Points)**

Consider a 2-sphere of unit radius with coordinates (θ,φ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2\,. \tag{5}$$

We are going to study the *geodesics* on this surface, which correspond to the trajectories which a free test particle would follow. One can think of the geodesics as the generalisation in curved space of straight lines in flat space. The geodesics can be found by solving the so-called *geodesic equation*. In the past exercise sheet we found out how this equation looks in flat space. In general, however, the geodesic equation takes the form

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{6}$$

where λ parametrizes the trajectories $x^\mu(\lambda)$.

(a) Compute the Christoffel symbols $\Gamma^\mu_{\rho\sigma}$ of the metric (5) using eq. (3).

(b) Show that lines of constant longitude (φ = constant) are geodesics, and that the line of constant latitude $\theta = \pi/2$ is also a geodesic, by showing that these trajectories fulfil equation (6). Note that in the first case you can take $\lambda = \theta$, while in the second $\lambda = \varphi$.

(c) In the Christoffel connection, a geodesic can also be defined as a path which *parallel transports* its own tangent vector. Parallel transport is a given way of moving a vector through curved space while keeping it constant. More precisely, the vector is transported in such a way that the covariant derivative along the direction of transport vanishes, eg.

$$v^\mu \nabla_\mu a^\nu = 0, \tag{7}$$

where a^ν is the vector we transport along the direction of v^μ.

Take a vector with components $a^\nu = (1,0)$ at $\theta = \theta_0, \varphi = 0$ and parallel transport it once around the circle of constant latitude $\theta = \theta_0$. In order to do this, solve the equations of motion you obtain from (7) for the components of a as a function of θ_0. Are lines of constant latitude, in general, geodesics?

(d) What is the modulus of a after the transport?

Exercise 3: Cosmological Redshift **(4 Points)**

The line element of the Robertson-Walker metric is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin(\theta)\, d\phi^2\right], \tag{8}$$

where $a(t)$ is the scale factor describing the expansion of the universe.
Due to this expansion, a light signal that is emitted with wavelength λ_1 at time $t = t_1$ will be observed with a different wavelength λ_0 at time $t = t_0$. The emitted and the observed wavelength are related by the *cosmological redshift* z

$$1 + z = \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)}\,. \tag{9}$$

A similar expression can be found for the momenta of massive particles. To see that, consider the geodesic equation written in terms of the four velocity u^μ:

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0\,. \tag{10}$$

a) Give the $\mu = 0$ component of the geodesic equation. You should have found in exercise 1 that the only non-vanishing component of $\Gamma^0_{\nu\rho}$ is $\Gamma^0_{ij} = \frac{\dot{a}(t)}{a(t)} g_{ij}$.

b) Next, show that

$$du^0 = \frac{|\vec{u}|}{u^0} d|\vec{u}|\,, \tag{11}$$

with $|\vec{u}| = g_{ij} u^i u^j$.
Then, recalling that $u^0 = \frac{dt}{d\tau}$, verify that the geodesic equation implies $|\vec{u}| \sim a^{-1}(t)$.