2. Übungsblatt zur Vorlesung **SS 2019**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 23. April 2019 16:00 Uhr

Exercise 1: The metric tensor in spherical coordinates **(6 Points)**

Consider a coordinate transformation in Minkowski space from cartesian to spherical space coordinates, $x^\mu = (t,x,y,z)^T \to \xi^\mu = (t,r,\theta,\varphi)^T$. The line element in the first set of coordinates reads $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$.

a) Compute the metric tensor, $g_{\mu\nu}$, in spherical space coordinates. Write down the line element in the new coordinates.

b) Compute the inverse metric tensor, $g^{\mu\nu}$, in both sets of coordinates.

c) Consider now Minkowski space restricted to the surface of a sphere with radius R. Use the previous results to compute the metric on this surface. Can you find a coordinate transformation that would convert this metric into its cartesian form, $ds^2|_{r=R} = -dt^2 + dx^2 + dy^2$?

Make your calculations explicit.

Exercise 2: The action of SR **(7 Points)**

The action of a free particle is minimized by the path with the shortest distance between two points. Since distances for timelike particles are measured by the proper time the action of a free particle in special relativity can be expressed in the following way:

$$S = \alpha \int \mathrm{d}\tau = \int L \mathrm{d}t, \tag{1}$$

where the proper time τ is defined by

$$d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu. \tag{2}$$

(a) Determine the relation between the proper time interval $\mathrm{d}\tau$ and the time interval $\mathrm{d}t$ by using eq. (2).

(b) Identify α by calculating the non-relativistic limit $v \ll 1$.

(c) Use the Euler-Lagrange equation to find the equation of motion for the relativistic case.

We now consider the general case, which yields the following action:

$$S = \alpha \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \alpha \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\mu}{d\tau}} \mathrm{d}\tau. \tag{3}$$

Here, $x^\mu = x^\mu(\tau)$.

d) Use the Euler-Lagrange equation again for the Lagrangian $L\left(x^\mu, \frac{dx^\mu}{d\tau}, \tau\right)$ in eq. (3) to obtain the equations of motion

$$\frac{d^2 x^\mu}{d\tau^2} = 0. \tag{4}$$

e) Finally, show that the equations of motion in a different coordinate system with coordinates y^μ are of the form

$$\frac{d^2 y^\mu}{d\tau^2} = -\frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial y^\nu \partial y^\rho}\frac{dy^\nu}{d\tau}\frac{dy^\rho}{d\tau}. \tag{5}$$

This equation is called the *geodesic equation*.

Exercise 3: Perfect Fluid **(7 Points)**

(a) Explain the term perfect fluid. What are the properties of the energy-momentum tensor $T^{\mu\nu}$ for a perfect fluid?

(b) Which important conservation law does the energy-momentum tensor satisfy? Calculate $\partial_\mu T^{\mu\nu}$ explicitly for a perfect fluid.

(c) Consider the projection tensor $P^\sigma{}_\nu = \delta^\sigma_\nu + u^\sigma u_\nu$. This tensor projects a vector onto a vector which is orthogonal to the fluid's four-velocity. Check whether $P^\sigma{}_\nu$ indeed is the projection tensor by calculating $P^\sigma{}_\nu P^\nu{}_\mu$ and $P^\sigma{}_\nu u^\nu$.

(d) Project the vector obtained in (b) onto a vector which is orthogonal to the fluid's four-velocity by using the projection tensor. What familiar equation from classical fluid mechanics do you find in the non-relativistic limit?