1. Übungsblatt zur Vorlesung **SS 2019**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 09. April 2019 16:00 Uhr

Exercise 1: Natural units **(2 Points)**

As you have seen in the lecture, natural units are defined by setting $c = \hbar = 1$. Use natural units to express 1 kg, 1 s and 1 m in powers of GeV. Use your results to express Newton's constant

$$G_{\mathrm{N}} = 6.674 \times 10^{-11} \, \frac{\mathrm{m}^3}{\mathrm{kg s}^2} \tag{1}$$

in natural units.

Exercise 2: Basics of Minkowski space **(6 Points)**

In special relativity time and euclidian three-dimensional space are unified in a four-dimensional vector space called Minkowski space. Spacetime events are described by *contravariant* 4-vectors

$$\left(x^\mu\right) = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}, \tag{2}$$

where t is the time coordinate, $\vec{x}$ is the position vector and $c = 1$ denotes the speed of light.

Additionally, *covariant* 4-vectors are defined as

$$x_\mu = \sum_\nu \eta_{\mu\nu} x^\nu \equiv \eta_{\mu\nu} x^\nu \quad \text{where} \quad \eta_{\mu\nu} = \begin{cases} -1, & \mu = \nu = 0 \\ +1, & \mu = \nu = 1,2,3 \\ 0, & \text{otherwise} \end{cases} . \tag{3}$$

We employ the *Einstein summation convention*: Whenever an index appears twice, summation over this index is implied.

The tensor $\eta_{\mu\nu}$ is the *metric* of Minkowski space; it can be used to map contravariant vectors onto covariant vectors and vice versa (it "lowers" and "raises" indices). Its inverse $\eta^{\mu\nu}$ is defined by $\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho$.

(a) Calculate or simplify (explicitly in terms of the components x^0, x^1, x^2, x^3) the following expressions:

  (i) $x_\nu = \eta_{\mu\nu} x^\mu$

  (ii) $\eta^\lambda{}_\lambda = \eta_{\mu\nu}\eta^{\mu\nu}$

  (iii) $\eta_{\alpha\beta}\eta^{\gamma\beta}$

  (iv) $\eta^{\mu\nu} x_\nu x_\mu$

  (v) $\eta^\mu{}_\alpha x_\sigma \eta^{\sigma\alpha} x_\mu$

(b) The scalar product of two 4-vectors x^μ and y^μ is given by the expression $x_\mu y^\mu$. Linear transformations $\Lambda^{\mu'}{}_\mu$ that map 4-vectors onto a new set of coordinates (labeled by μ') and leave the scalar product $x_\mu y^\mu$ invariant are called *Lorentz transformations*. Contravariant 4-vectors transform according to

$$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu . \tag{4}$$

Derive the respective transformation law for covariant 4-vectors. How does the derivative $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ transform? (*Hint: Chain rule*)

Exercise 3: Tensor properties **(6 Points)**

Let $S_{\mu\nu} = S_{\nu\mu}$ be a symmetric tensor; $A_{\mu\nu} = -A_{\nu\mu}$ an antisymmetric one. Let $T_{\mu\nu}$ be an additional arbitrary tensor of rank 2. Arbitrary tensors $T_{\mu_1\mu_2\ldots\mu_n}$ of rank n can be symmetrized or antisymmetrized according to

$$T_{(\mu_1\mu_2\ldots\mu_n)} := \frac{1}{n!}\sum_P T_{\mu_1\mu_2\ldots\mu_n} \tag{5}$$

and

$$T_{[\mu_1\mu_2\ldots\mu_n]} := \frac{1}{n!}\sum_P \mathrm{sgn}(P)\, T_{\mu_1\mu_2\ldots\mu_n}, \tag{6}$$

respectively. Here, P denotes the permutations of the indices μ_i and $\mathrm{sgn}(P)$ is the sign of the permutation, defined by $\mathrm{sgn}(P) = \begin{cases} +1, & \text{if the number of permutations in } P \text{ is even} \\ -1, & \text{if the number of permutations in } P \text{ is odd} \end{cases}$.

(a) Show explicitly:

$$S_{\mu\nu}T^{\mu\nu} = S_{\mu\nu}T^{(\mu\nu)}, \quad A_{\mu\nu}T^{\mu\nu} = A_{\mu\nu}T^{[\mu\nu]}, \quad S_{\mu\nu}A^{\mu\nu} = 0. \tag{7}$$

(b) Show that an arbitrary rank-2 tensor can be decomposed into a symmetric and an antisymmetric part:

$$T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}. \tag{8}$$

Can this also be done for tensors of rank $n > 2$? Provide a proof or a counterexample.

(c) Show that in general

$$T^{\mu}{}_{\nu} \neq T_{\nu}{}^{\mu}. \tag{9}$$

Exercise 4: Lagrangian formalism and Maxwell's equations **(6 Points)**

The lagrangian density of the free electromagnetic field is given by

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{10}$$

where the electromagnetic field tensor is defined in terms of electromagnetic four-potential $(A^\mu) = (\phi, \vec{A})^{\mathrm{T}}$ as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The Euler-Lagrange (EL) equations follow from the minimum action principle which states that the action, defined as $S = \int d^4x\mathscr{L}$ is stationary, that is

$$\delta S = 0, \tag{11}$$

and are given as:

$$\frac{\partial\mathscr{L}}{\partial A_\nu} - \partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu)} = 0. \tag{12}$$

(a) Using the EL equations derive the equations of motion for the electromagnetic potential A_μ.

(b) Write down the resulting equations in terms of the electric and magnetic fields which are components of $F_{\mu\nu}$, that is

$$F^{0i} = E^i, \quad F^{ij} = \epsilon^{ijk} B_k, \tag{13}$$

where indices i, j refer to the corresponding spatial components, $i, j = 1, 2, 3$ and ϵ^{ijk} is totally antisymmetric tensor with respect to exchanges of any two indices (Levi-Civita tensor), with the convention $\epsilon^{123} = 1$.

(c) Show that in the Lorentz gauge the equations of motion derived in (a) reduce to the wave equation.