7. Übungsblatt zur Vorlesung **SS 2016**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 12. Juli 2016 16:00 Uhr

Exercise 1: Gravitational lensing **(10 Points)**

Consider a photon travelling along a lightlike trajectory in the gravitational field of a spherical mass M. The coordinates can be chosen in such a way that the trajectory lies in the equatorial plane ($\theta = \pi/2$). The Schwarzschild metric is given as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1}$$

(a) Draw a sketch of the situation.

(b) Show that the conserved quantities associated with the symmetries of the Schwarzschild metric are given by

$$e \equiv \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \quad \text{and} \quad l \equiv r^2 \frac{d\phi}{d\tau}. \tag{2}$$

(c) Derive the photon's orbital equation

$$\left(\frac{dv}{d\phi}\right)^2 = \frac{e^2}{l^2} - v^2(1 - 2GMv), \tag{3}$$

where $v \equiv 1/r$.

(d) Use the orbital equation to obtain the following second order differential equation:

$$\frac{d^2 v}{d\phi^2} + v = 3GMv^2. \tag{4}$$

(e) For the initial conditions $v(0) = 0$ and $v'(0) = \frac{1}{b}$, show that the general solution of the unperturbed version of equation (4) ($M = 0$) reads

$$v(\phi) = \frac{\sin\phi}{b}. \tag{5}$$

What is the physical meaning of the parameter b?

(f) Solve equation (4) to leading order in GM. In order to do so, take the solution (5) of the unperturbed differential equation, plug it into the right hand side of equation (4) and solve the resulting inhomogenous differential equation.
If you don't know how to solve it, show that

$$v(\phi) = \frac{\sin\phi}{b} + \frac{GM}{b^2}(1 - \cos\phi)^2 \tag{6}$$

solves (4) to leading order in GM.

(g) Show that the deflection angle α of an incident light ray is given by

$$\alpha = \frac{4GM}{b} \tag{7}$$

to leading order in $\frac{GM}{b}$.

Exercise 2: Black holes **(10 Points)**

(a) Use the Schwarzschild metric in spherical coordinates (1) and your knowledge about the four-velocity U^μ to expand $U^\mu U_\mu$.

(b) Multiply your result by $(1-2GM/r)$ and use equation (2) to derive a differential equation for $r(\tau)$.

(c) Your equation should be of the form

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V(r) = \frac{1}{2}e^2. \tag{8}$$

This is the equation for a classical particle of *unit* mass. Take the limit $r \to \infty$. Which familiar formular do you recognize?

(d) Draw the light cones for the Schwarzschild metric in spherical coordinates in the $t-r$-plane for three different values of r. What do you recognize, as the light cone approaches the Schwarzschild radius $R_\mathrm{s} = 2GM$?

(e) Apply the following coordinate transformation

$$r' = r + 2GM \ln\left(\frac{r}{2GM-1}\right) \tag{9}$$

to the Schwarzschild metric. What are the advantages and drawbacks of this coordinate system? *Hint: What are the coordinates of the Schwarzschild radius after the coordinate transformation?*

(f) Uncharged rotating black holes can be described by the Kerr metric. The Kerr metric in Boyer-Lindquist coordinates reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\rho^2}{\Delta^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2+a^2\right)\sin^2\theta\mathrm{d}\phi^2 + \frac{2GMr}{\rho^2}\left(a\sin^2\theta\mathrm{d}\phi - \mathrm{d}t\right)^2 \tag{10}$$

with

$$\Delta(r) = r^2 - 2GMr + a^2 \tag{11}$$

and

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta. \tag{12}$$

The constant a measures the rotation of the black hole. What are the singularities of the radial component g_{rr} and the corresponding event horizons? Which horizon is physically significant?

(g) Derive the radius of the event horizons for $g_{tt} = 0$.

(h) Where are the intersections of the physically relevant horizons?

(i) Calculate the limit $a \to 0$. What do you recognize?

Exercise 3: Short questions **(10 extra Points)**

(a) How does the tensor $T^{\mu\ \ \rho\sigma}_{\ \ \nu\ \ \ \ \lambda}$ transform under coordinate transformations $x^{\mu} \to x^{\mu'}$?

(b) What is a geodesic?

(c) What is the origin of the conservation of the energy momentum tensor $\nabla_{\mu} T^{\mu\nu} = 0$?

(d) Explain how the Riemann tensor encodes information about the curvature of a manifold.

(e) What are Killing vectors?

(f) Explain the difference between a static and a stationary metric. Which examplary astronomical objects can be described by a static and a stationary metric?

(g) Explain the physical relavance of coordinate singularities and a singularity of the curvature.

(h) Name a coordinate system which allows you to describe the entire spacetime corresponding to the Schwarzschild metric.

(i) Explain Birkhoff's theorem in your own words.

(j) What is the action of general (special) relativity?