5. Übungsblatt zur Vorlesung **SS 2016**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 14. Juni 2016 16:00 Uhr

Exercise 1: Fundamentals of cosmology **(15 Points)**

Let $T_{\mu\nu}$ be the energy-momentum tensor of a perfect fluid, given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{1}$$

where ρ is the energy density, p is the isotropic pressure in the fluid's rest-frame and u^μ is its four-velocity. Furthermore, the Friedmann–Lemaître–Robertson–Walker (FLRW) metric is given as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]. \tag{2}$$

You can find the Christoffel symbols and the Ricci tensor on the next page. The equations that relate the evolution of the scale factor $a(t)$ to the matter content of the universe are called *Friedmann equations*.

(a) Calculate the Ricci scalar R.

(b) Derive the Friedmann equations.
The first Friedmann equation can be obtained from the 00-component of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3}$$

Use the trace of (3) and the first equation and to find the second Friedmann equation.

(c) Consider $\nabla_\mu T^{\mu 0} = 0$ to derive a continuity equation for ρ and p. Is this equation independent of the Friedmann equations?

(d) Suppose that pressure and energy density fulfill the equation of state

$$p = w\rho, \tag{4}$$

where w is a time-independent constant. Use this equation and the continuity equation from (c) to express the energy density in terms of the scale factor a.

(e) Determine the time dependence of the scale factor $a = a(t)$ using the Friedmann equations and $\rho(a)$ for the case $k = 0$.

(f) Consider the energy densities of dust, photons and vacuum energy. Find the constant w for each case. How do the respective energy densities depend on the scale factor a? How does the scale factor evolve in time, if the universe is dominated by dust, photons or vacuum energy?

Exercise 2: Properties of the Einstein equations **(5 Points)**

With the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ the field equations of general relativity can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{5}$$

Since both $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric, (5) represents 10 independent equations. The state of a system at a given time can be described by the values of the metric $g_{\mu\nu}$ and the time derivative $\partial_t g_{\mu\nu}$ – analogously to a state consisting of coordinates x^i and momenta p^i in classical mechanics. The evolution of such a state is then governed by Einstein's equations (5).

Show that of the 10 independent equations only 6 describe the dynamical evolution of the state, while the 4 equations given by $G^{0\nu} = 8\pi G T^{0\nu}$ merely serve as initial constraints. In order to do so, use the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0 \tag{6}$$

to show that the expression $\partial_t G^{0\nu}$ contains no third-order time derivatives and thus no second-order time derivatives occur in $G^{0\nu}$. Discuss.

Appendix

You have already calculated the non-vanishing Christoffel symbols of the FLRW metric on sheet 3. They are given by

$$\begin{aligned}
&\Gamma^t_{rr} = \frac{a\dot{a}}{1-kr^2}, && \Gamma^t_{\theta\theta} = a\dot{a}r^2, && \Gamma^t_{\phi\phi} = a\dot{a}r^2\sin^2\theta, \\
&\Gamma^r_{rr} = \frac{kr}{1-kr^2}, && \Gamma^r_{\theta\theta} = -r(1-kr^2), && \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta, \\
&\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, && \Gamma^\phi_{\theta\phi} = \cot\theta \\
&\Gamma^r_{tr} = \Gamma^\theta_{t\theta} = \Gamma^\phi_{t\phi} = \frac{\dot{a}}{a},
\end{aligned} \tag{7}$$

or related to these by symmetry. The nonzero components of the Ricci tensor are

$$\begin{aligned}
&R_{tt} = -3\frac{\ddot{a}}{a}, && R_{rr} = \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2}, \\
&R_{\theta\theta} = r^2(a\ddot{a}+2\dot{a}^2+2k), && R_{\phi\phi} = r^2(a\ddot{a}+2\dot{a}^2+2k)\sin^2\theta.
\end{aligned} \tag{8}$$