4. Übungsblatt zur Vorlesung **SS 2016**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 31. Mai 2016 16:00 Uhr

Exercise 1: Symmetries of the Curvature Tensor **(6 Points)**
Show the following symmetry relations of the curvature tensor $R_{\mu\nu\rho\lambda}$:

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda} \tag{1}$$
$$R^{\mu}{}_{\nu\rho\lambda} = -R^{\mu}{}_{\nu\lambda\rho} \tag{2}$$
$$R_{\mu\nu\rho\lambda} = R_{\rho\lambda\nu\mu} \tag{3}$$
$$R^{\mu}{}_{\nu\rho\lambda} + R^{\mu}{}_{\lambda\nu\rho} + R^{\mu}{}_{\rho\lambda\nu} = 0 \tag{4}$$

To show equations (1) and (3) it is useful to employ Riemann normal coordinates at the point p. This coordinate system satisfies the following properties at the point p:

$$\Gamma^{\mu}_{\alpha\beta} = 0, \quad \partial_\nu \Gamma^{\mu}_{\alpha\beta} \neq 0 \quad \text{and} \quad \partial_\mu g^{\alpha\beta} = 0.$$

Exercise 2: Robertson-Walker Metric and Photons **(6 Points)**
In cartesian coordinates, the Robertson-Walker metric for flat space is given by:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{5}$$

(a) In analogy to particles with mass, the 4-momentum of a massless particle can be written as

$$P^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \quad \text{with} \quad P^\alpha P_\alpha = 0 \quad \text{and} \quad P^0 = E. \tag{6}$$

Since we are dealing with a massless particle, the affine parameter λ cannot be the proper time. Use the geodesic equation to show that the energy of the paricle obeys the following relation

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\dot{a}}{a} E = 0. \tag{7}$$

Hint: You do not have to calculate all Christoffel symbols.

(b) Show that $E \sim \frac{1}{a}$ holds. What is the physical meaning of this result?

Exercise 3: Curvature of the Torus **(8 Points)**
The torus can be embedded in three dimensional space with the parametrization

$$\vec{r}(\theta, \varphi) = \begin{pmatrix} \cos\theta (a + r\cos\varphi) \\ \sin\theta (a + r\cos\varphi) \\ r\sin\varphi \end{pmatrix}, \tag{8}$$

where a and r are constants.

(a) Show that the induced metric tensor is given by $g_{\mu\nu} = \mathrm{diag}\big((a + r\cos\varphi)^2, r^2\big)$.

(b) Determine the geodesic equation starting from

$$\delta \int \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda = 0, \tag{9}$$

and read off the non-vanishing Christoffel symbols.

(c) Calculate the non-vanishing components of the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \tag{10}$$

Hint: Use the relations from exercise 1 and think about how many independent components there are!

(d) Calculate the Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and the Ricci scalar $R = R^\mu{}_\mu$.

(e) The torus can also be embedded in four dimensional space using the parametrization

$$\vec{r}(u, v) = \begin{pmatrix} A\cos u \\ A\sin u \\ B\cos v \\ B\sin v \end{pmatrix}. \tag{11}$$

Calculate the components of the metric tensor and discuss your results.