3. Übungsblatt zur Vorlesung **SS 2016**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 17. Mai 2016 16:00 Uhr

Exercise 1: The Friedmann–Lemaître–Robertson–Walker metric **(9 Points)**
In spherical coordinates, the FLRW metric is given as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi\right], \tag{1}$$

where, $a(t)$ is the time-dependent scale factor and $k \in \{-1, 0, +1\}$ is a parameter which determines the curvature of spacetime.
Calculate all non-vanishing Christoffel symbols $\Gamma^{\mu}_{\alpha\beta}$ for the FLRW metric (1) using the relation

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}\left(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}\right). \tag{2}$$

Exercise 2: The scalar theory of gravity **(5 Points)**
Newton's gravitational law is expressed in the differential form

$$\nabla^2 \Phi = 4\pi G\rho, \tag{3}$$

where ρ is the rest mass energy density and G is the gravitational constant. This law is not Lorentz invariant and expresses "action at a distance", due to the lack of a time derivative. One could attempt to build a relativistic theory of gravity by making the above equation Lorentz invariant as

$$\Box \Phi = 4\pi G T^{\mu}{}_{\mu}, \tag{4}$$

where $T_{\mu\nu}$ is the energy momentum tensor. Calculate $T^{\mu}{}_{\mu}$ explicitly for the energy momentum tensor of the electromagnetic field, given as

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right], \tag{5}$$

and explain why equation (4) cannot be a correct description of gravity.

Exercise 3: Earth-based tests of GR **(6 Points)**

The effect of the gravitational redshift is already visible in earth laboratories. Isotopes of an iron atom located at the top of a building can only be stimulated by the light emission of iron atoms at the bottom of the buidling if the relative vertical speed of the atoms compensates the gravitational redshift.

(a) Calculate the compensation speed v in SI-units for a building with height $\Delta h = 30\,\text{m}$ and neglect general relativity corrections in the calculation of the Doppler effect.

(b) Nowadays it is possible to measure the effect of general relativity for height differences as small as 1 cm on the surface of the earth. Recalculate the compensation speed for $\Delta h = 1\,\text{cm}$. What do you notice?

(c) Modern optical atomic clocks are so precise that they are influenced by differences in the gravitational potential. What is the precision of an atomic clock in terms of $\frac{\Delta\tau}{t}$ if the clock is sensitive to 1 cm differences in the altitude? *Hint: To avoid numerical problems, use the following Taylor expansion:*

$$\sqrt{1-\frac{b}{a+x}} = \sqrt{1-\frac{b}{a}} + \frac{bx}{2a^2\sqrt{1-\frac{b}{a}}} + \mathcal{O}\left(x^2\right) \tag{6}$$

(d) Argue if it is possible or not to measure the absolute gravitational potential with atomic clocks.