2. Übungsblatt zur Vorlesung **SS 2016**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 3. Mai 2016 16:00 Uhr

Exercise 1: Maxwell's equations **(7 Points)**
The lagrangian density of the free electromagnetic field is given by

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

where the electromagnetic field tensor is defined in terms of electromagnetic four-potential $(A^\mu) = (\phi, \vec{A})^\mathrm{T}$ as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The Euler-Lagrange (EL) equations follow from the minimum action principle which states that the action, defined as $S = \int d^4x \mathscr{L}$ is stationary, that is

$$\delta S = 0, \tag{2}$$

and are given as:

$$\frac{\partial \mathscr{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} = 0. \tag{3}$$

(a) Using the EL equations derive the equation of motion for the electromagnetic potential A_μ.

(b) Write down the resulting equation in terms of the electric and magnetic fields which are components of $F_{\mu\nu}$, that is

$$F^{0i} = E^i, \quad F^{ij} = \epsilon^{ijk} B_k, \tag{4}$$

where indices i, j refer to the corresponding spatial components, $i, j = 1, 2, 3$ and ϵ^{ijk} is totally antisymmetric tensor with respect to exchanges of any two indices (Levi-Civita tensor), with the convention $\epsilon^{123} = 1$.

(c) Where does the rest of the Maxwell equations come from? Write down the remaining Maxwell equations in terms of $\vec{E}$ and $\vec{B}$.

(d) Solve the equation of motion for A_μ.
Hint: To simplify the equation of motion for A_μ use the "gauge invariance" of the Maxwell lagrangian, that is invariance under the following transformation of A_μ:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x), \tag{5}$$

where $\alpha(x)$ is an arbitrary differentiable function.

Exercise 2: Boosted Lorentz Force **(5 Points)**
The four-force $f^\mu \equiv \frac{dp^\mu}{d\tau}$ acting on a test charge q with four-velocity u^μ is given as

$$f^\mu = q F^{\mu\nu} u_\nu, \tag{6}$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor.

(a) Calculate f^μ explicitly and show that you obtain the well-known Lorentz-force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{7}$$

from the spatial components in the non-relativistic limit. What is the meaning of the time component?

(b) Assume that the particle moves along the x-axis with velocity v (i.e. $u^1 = \gamma v$, all other spatial components vanish), and is influenced by a magnetic field of strength B along the z-axis. Calculate the resulting Lorentz force.

(c) Now, perform a boost into the particle's rest frame with a Lorentz transformation Λ that fulfills

$$\Lambda^{\mu'}{}_{\mu} u^\mu = u^{\mu'} = \begin{cases} 1, & \mu' = 0 \\ 0, & \text{otherwise} \end{cases} . \tag{8}$$

Calculate the components of the field strength tensor as well as the resulting Lorentz force in this frame.

Exercise 3: Perfect Fluid **(4 Points)**

(a) Explain the term perfect fluid. What are the properties of the energy momentum tensor $T^{\mu\nu}$ for a perfect fluid?

(b) Which important conservation law does the energy-momentum tensor satisfy? Calculate $\partial_\mu T^{\mu\nu}$ explicitly for a perfect fluid.

(c) Project the resulting vector onto a vector which is orthogonal to the fluid's four-velocity by using the projection tensor $P^\sigma{}_\nu = \delta^\sigma_\nu + u^\sigma u_\nu$. What familiar equation from classical fluid mechanics do you find in the non-relativistic limit?

Exercise 4: The action of SR **(4 Points)**

The action of a free particle in flat space will always be minimized by the path with the shortest distance between two points. Since the distances for timelike particles are measured by the proper time the action of a free particle in special relativity can be expressed in the following way:

$$S = \alpha \int \mathrm{d}\tau = \int L \mathrm{d}t \tag{9}$$

(a) Determine the Lagrangian L with respect to the unknown constant α.

(b) Identify α by calculating the non-relativistic limit $v \ll 1$.

(c) Use the Euler-Lagrange equation to find the equation of motion for the relativistic case.