1. Übungsblatt zur Vorlesung **SS 2016**
Allgemeine Relativitätstheorie **Prof. G. Hiller**
Abgabe: bis Dienstag, den 26. April 2016 16:00 Uhr

Exercise 1: Minkowski space **(7 Points)**
In special relativity time and euclidian three-dimensional space are unified in a four-dimensional vector space called Minkowski space. Spacetime events are described by *contravariant* 4-vectors

$$\left(x^\mu\right) = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}, \tag{1}$$

where t is the time coordinate, $\vec{x}$ is the position vector and $c = 1$ denotes the speed of light.
Additionally, *covariant* 4-vectors are defined as

$$x_\mu = \sum_\nu \eta_{\mu\nu} x^\nu \equiv \eta_{\mu\nu} x^\nu \quad \text{where} \quad \eta_{\mu\nu} = \begin{cases} -1, & \mu = \nu = 0 \\ +1, & \mu = \nu = 1,2,3 \\ 0, & \text{otherwise} \end{cases} . \tag{2}$$

We employ the *Einstein summation convention*: Whenever an index appears twice, summation over this index is implied.
The tensor $\eta_{\mu\nu}$ is the *metric* of Minkowski space; it can be used to map contravariant vectors onto covariant vectors and vice versa (it "lowers" and "raises" indices). Its inverse $\eta^{\mu\nu}$ is defined by $\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho$.

(a) Calculate or simplify (explicitly in terms of the components x^0, x^1, x^2, x^3) the following expressions:

  (i) $x_\nu = \eta_{\mu\nu} x^\mu$
  (ii) $\eta^\lambda{}_\lambda = \eta_{\mu\nu}\eta^{\mu\nu}$
  (iii) $\eta_{\alpha\beta}\eta^{\gamma\beta}$
  (iv) $\eta^{\mu\nu} x_\nu x_\mu$
  (v) $\eta^\mu{}_\alpha x_\sigma \eta^{\sigma\alpha} x_\mu$

(b) The scalar product of two 4-vectors x^μ and y^μ is given by the expression $x_\mu y^\mu$. Linear transformations $\Lambda^{\mu'}{}_\mu$ that map 4-vectors onto a new set of coordinates (labeled by μ') and leave the scalar product $x_\mu y^\mu$ invariant are called *Lorentz transformations*. Contravariant 4-vectors transform according to

$$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu. \tag{3}$$

Derive the respective transformation law for covariant 4-vectors. How does the derivative $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ transform? (*Hint: Chain rule*)

Exercise 2: Tensor properties **(6 Points)**

Let $S_{\mu\nu} = S_{\nu\mu}$ be a symmetric tensor; $A_{\mu\nu} = -A_{\nu\mu}$ an antisymmetric one. Let $T_{\mu\nu}$ be an additional arbitrary tensor of rank 2. Arbitrary tensors $T_{\mu_1\mu_2\ldots\mu_n}$ of rank n can be symmetrized or antisymmetrized according to

$$T_{(\mu_1\mu_2\ldots\mu_n)} := \frac{1}{n!}\sum_P T_{\mu_1\mu_2\ldots\mu_n} \tag{4}$$

and

$$T_{[\mu_1\mu_2\ldots\mu_n]} := \frac{1}{n!}\sum_P \mathrm{sgn}(P)\, T_{\mu_1\mu_2\ldots\mu_n}, \tag{5}$$

respectively. Here, P denotes the permutations of the indices μ_i and $\mathrm{sgn}(P)$ is the sign of the permutation, defined by $\mathrm{sgn}(P) = \begin{cases} +1, & \text{if } P \text{ is even} \\ -1, & \text{if } P \text{ is odd} \end{cases}$.

(a) Show explicitly:

$$S_{\mu\nu}T^{\mu\nu} = S_{\mu\nu}T^{(\mu\nu)}, \quad A_{\mu\nu}T^{\mu\nu} = A_{\mu\nu}T^{[\mu\nu]}, \quad S_{\mu\nu}A^{\mu\nu} = 0. \tag{6}$$

(b) Show that an arbitrary rank-2 tensor can be decomposed into a symmetric and an antisymmetric part:

$$T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}. \tag{7}$$

Can this also be done for tensors of rank $n > 2$? Provide a proof or a counterexample.

(c) Show that in general

$$T^{\mu}{}_{\nu} \neq T_{\nu}{}^{\mu}. \tag{8}$$

Exercise 3: Light cone, proper time and 4-velocity **(7 Points)**

The norm of a 4-vector x^μ is defined as

$$x^2 \equiv x_\mu x^\mu = \eta_{\mu\nu}x^\mu x^\nu. \tag{9}$$

(a) Describe the physical meaning of the following three cases. Sketch your results in an $|\vec{x}|$–t-diagram.

(i) $x_\mu x^\mu = 0$

(ii) $x_\mu x^\mu < 0$

(iii) $x_\mu x^\mu > 0$

For trajectories $x^\mu(\lambda)$ that obey $x_\mu x^\mu < 0$ we can define their *proper time* τ by

$$d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu. \tag{10}$$

(b) Show that the differential proper time $d\tau$ is related to the differential coordinate time dt via $d\tau = dt/\gamma$ and derive an explicit expression for γ. Which values can $1/\gamma$ take?

(c) Calculate the norm of the 4-velocity $u^\mu \equiv \dfrac{dx^\mu}{d\tau}$.