13. Übungsblatt zur Vorlesung — WS 2019/20

Einführung in die Elementarteilchentheorie — **Prof. G. Hiller**
Abgabe: bis Di. 21. Januar 2020, 12:00 Uhr — **Kasten 245**
http://people.het.physik.tu-dortmund.de/~ghiller/WS1920ETT.html

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Exercise 1: Spontaneous symmetry breaking **(5 Points)**

The Lagrangian

$$\mathscr{L} = \frac{1}{2}\left(\partial_\nu \phi_i\right)^2 + \frac{1}{2}\mu^2 \phi_i^2 - \frac{1}{4}\lambda\left(\phi_i^2\right)^2 , \quad i = 1,2,3 \text{ (summation convention)} \tag{1.1}$$

describes a scalar theory with a global $O(3)$–symmetry under which a field $\phi = \left(\phi_1, \phi_2, \phi_3\right)$ transforms like a vector. Here, $\mu^2 > 0$ and $\lambda > 0$.

a) Break the symmetry down to an $O(2)$–symmetry by introducing a suitable vacuum expectation value (VEV) $\langle\phi\rangle$ and showing that one massive particle as well as two massless *Goldstone* bosons are generated. Find an expression for the mass of the particles as a function of the parameters μ and λ.

b) Show all possible particle interactions for the (spontaneously) broken symmetry by drawing the corresponding vertices.

Exercise 2: GSW theory: $SU(2) \times U(1)$ **(8 Points)**

The elements of the gauge group $SU(2) \times U(1)$ can be written as

$$U(x) = \exp\left(\mathrm{i}\,\alpha^a(x)\,t^a\right)\exp\left(\mathrm{i}\frac{\beta(x)}{2}\right), \tag{2.1}$$

where the $SU(2)$ generators are given by the Pauli matrices $\sigma^a = 2\,t^a$. We now want to break the gauge symmetry of the GSW theory by choosing a ground state ϕ_0 for a scalar doublet field ϕ,

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}. \tag{2.2}$$

a) Show that the ground state is invariant under the gauge transformation

$$U(x) = \exp\left(\mathrm{i}\,\alpha^3(x)\,t^3\right)\exp\left(\mathrm{i}\frac{\beta(x)}{2}\right) \tag{2.3}$$

for specific phases $\alpha^3(x)$ and $\beta(x)$. What relation needs to hold for these two phases?

The covariant derivative in the GSW theory reads

$$D_\mu = \partial_\mu - \mathrm{i}\,g\,A_\mu^a\,t^a - \mathrm{i}\,g'\,Y\,B_\mu, \tag{2.4}$$

where A^a_μ and B_μ denote the gauge fields of $SU(2)$ and $U(1)$, respectively. The corresponding mass eigenstates are

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \left(A^1_\mu \mp \mathrm{i}\, A^2_\mu \right), \tag{2.5}$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g\, A^3_\mu - g'\, B_\mu \right), \tag{2.6}$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g'\, A^3_\mu + g\, B_\mu \right). \tag{2.7}$$

b) Show that the masses of the fields $W^\pm$, Z^0_μ and A_μ are given by

$$M_W = g\,\frac{v}{2}, \quad M_Z = \frac{v}{2}\sqrt{g^2 + g'^2}, \quad M_A = 0. \tag{2.8}$$

c) The weak mixing angle θ_w relates the couplings g and g':

$$\cos\theta_w = \frac{g}{\sqrt{g^2 + g'^2}} \qquad \sin\theta_w = \frac{g'}{\sqrt{g^2 + g'^2}}. \tag{2.9}$$

Show that the following mixing relation holds:

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix}. \tag{2.10}$$

d) How would one measure the mixing angle θ_w?

e) Explain how the quark masses $SU(2) \times U(1)$ break the gauge structure.

f) How large is the Yukawa coupling of the top quark?

Exercise 3: Minimum of scalar potentials and scalar masses (7 Points)

We consider a system with two real, scalar fields ϕ_1 ,ϕ_2 and the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{i=1,2} \left(\partial_\mu \phi_i\right)\left(\partial^\mu \phi_i\right) - V\left(\phi_1, \phi_2\right), \tag{3.1}$$

with the potential

$$V\left(\phi_1, \phi_2\right) = \frac{1}{2}\mu_1^2 \phi_1^2 + \frac{1}{2}\mu_2^2 \phi_2^2 - b\,\phi_1\,\phi_2 + \frac{g^2}{8}\left(\phi_2^2 - \phi_1^2\right)^2, \tag{3.2}$$

where μ_1^2, μ_2^2, b and g are real parameters and $b > 0$[1].

a) Which symmetries are present in the potential V for $b = 0$ and $B \neq 0$?

[1] This model is borrowed from the Higgs sector of the Minimal Supersymmetric Standard Model (MSSM).

b) Discuss properties of the potential V:
Show that for spontaneous symmetry breaking (SSB) the relation

$$b^2 > \mu_1^2 \mu_2^2 , \tag{3.3}$$

(where $\phi_1 = \phi_2 = 0$ is *not* a stable solution) holds, as well as

$$2b^2 < \mu_1^2 + \mu_2^2 . \tag{3.4}$$

For $|\phi_1| = |\phi_2|$ the potential is supposed to be bounded from below, that is $V > 0$ for $|\phi_1|, |\phi_2| \to \infty$.

c) Minimise the potential V and state the equations for the VEVs v_1 and v_2 of ϕ_1 and ϕ_2 at the minimum, where $v_1 = v \cos\beta$ and $v_2 = v \sin\beta$. The explicit solution of these equations is not part of this exercise.

d) Write down the Lagrangian after SSB, i.e. $\phi_1 = v_1 + h_1$, $\phi_2 = v_2 + h_2$ and consider terms up to second order in the Higgs fields h_1 and h_2. These terms are the mass terms, which can be conveniently written in matrix form

$$\mathscr{L}_{\text{mass}} = -\frac{1}{2} \begin{pmatrix} h_1 & h_2 \end{pmatrix} M^2 \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} . \tag{3.5}$$

Calculate the mass matrix M^2 and then perform a diagonalisation to obtain the mass eigenstates H_1 and H_2 as well as their masses M_1 and M_2.