11. Übungsblatt zur Vorlesung — WS 2019/20

Einführung in die Elementarteilchentheorie — **Prof. G. Hiller**

Abgabe: bis Di. 07. Januar 2020, 12:00 Uhr — **Kasten 245**

http://people.het.physik.tu-dortmund.de/~ghiller/WS1920ETT.html

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Exercise 1: Short questions **(6 Points)**

a) What is Fermis Golden Rule?

b) What is the optical theorem?

c) What is the S–matrix and what is its role in high energy physics?

d) Write down the Klein-Gordon equation, Maxwell's equations, and the Dirac equation with their corresponding lagrangian densities in covariant notation. What is the fundamental difference between the particles that are described by these three equations?

e) What is the motivation for introducing quantum fields?

f) Write down the quantized version of the Klein-Gordon field, the Dirac field and the photon field.

g) Wick's theorem for real scalar fields says that for an arbitrary number n

$$T\left\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right\} = N\left\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right\} + \text{ all possible contractions}, \quad (1.1)$$

where T is the time ordered and N is the normal ordered product. Take this example and explain schematically the connection between Wick's theorem and Feynman diagrams.

h) Explain the physical reason for the appearance of spin sums in the calculation of cross sections.

i) Why is a mass term $M^2 A_\mu A^\mu$ forbidden in the QED lagrangian?

j) What is the Ward identity and what does it imply for photons?

k) Draw all Feynman diagrams that contribute to Bhabha scattering $e^+e^- \to e^+e^-$ to lowest non-trivial order.

l) Which conservation law is implied by $U(1)$ gauge invariance?

Exercise 2: Local gauge invariance for non-abelian symmetry groups **(8 Points)**

Local gauge transformations of fermion fields $\Psi = (\Psi_1, \ldots, \Psi_N)^T$ under the elements g of a group $G = SU(N)$ are given by unitary transformations U_g in the space of fields:

$$\Psi'_i(x) = \sum_{j=1}^{N} (U_g(x))_{ij}\, \Psi_j(x)\,, \quad (2.1)$$

with

$$(U_g(x))_{ij} = \exp\left[\mathrm{i}\sum_{a=1}^{N^2-1} \phi^a(x)\, T^a_{ij}\right] \approx \delta_{ij} + \mathrm{i}\sum_{a=1}^{N^2-1} \phi^a(x)\, T^a_{ij}\,. \tag{2.2}$$

The $\phi^a(x)$ are the local parameters of the transformation and the T^a_{ij} the generators of the group G in the fundamental representation. In the following we make use of the sum convention. Remember that the generators T^a are part of the group algebra $[T^a, T^b] = \mathrm{i} f^{abc} T^c$ with antisymmetric structure constants f^{abc}. A group is called non-abelian, whenever the structure constant is non-zero for one combination of a, b, c.
Local gauge invariance now requires the invariance of the lagrangian density under the gauge transformation of the fermion fields Ψ given above. This implies the need to introduce massless gauge fields G^a_μ in the covariant derivative $D_\mu = \partial_\mu + \mathrm{i} g G^a_\mu T^a$. This covariant derivative transforms like

$$D'_\mu = U_g\, D_\mu\, U_g^{-1}\,. \tag{2.3}$$

Analogously to abelian QED, one can introduce an antisymmetric field strength tensor, however there is an additional term appearing in its definition

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g\, f^{abc} G^b_\mu G^c_\nu\,. \tag{2.4}$$

a) Prove that $[D_\mu, D_\nu] = \mathrm{i}\, g\, G^a_{\mu\nu} T^a$.

b) Use the transformation rule for D_μ, in order to get the transformation rule for $G^a_{\mu\nu}$.

c) Use b) to show that a kinetic term for non-abelian gauge fields $\mathscr{L}_{\mathrm{kin}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$ is indeed gauge invariant.

Exercise 3: Running couplings **(6 Points)**
The coupling constants α_i in gauge theories are momentum and scale dependent. The running of these couplings from a scale μ_0 to a scale μ is governed by the so called renormalization group equation. To lowest order in perturbation theory it is given by

$$\mu \frac{\mathrm{d}\alpha_i}{\mathrm{d}\mu}(\mu) = -\frac{\beta_0^{(i)}}{2\pi}\, \alpha_i^1(\mu)\,. \tag{3.1}$$

The coefficients $\beta_0^{(i)}$ in strong interaction ($i = s$, QCD) and in electromagnetism ($i = e$, QED) follow from loop calculations and are given by:

$$\beta_0^{(s)} = \frac{11}{3} N_c - \frac{2}{3} N_f\,, \tag{3.2}$$

$$\beta_0^{(e)} = -\frac{4}{3}\left(Q_u^2 N_c N_u + Q_d^2 N_c N_d + Q_l^2 N_l\right). \tag{3.3}$$

Here, $N_c = 3$ is the number of color charges of the quarks under QCD and N_f the number of active quarks. N_u and N_d are the number of active quarks with electric charge $Q_u = 2/3$, $Q_d = -1/3$, respectively. The number of active leptons with $Q_l = -1$ is N_l.
Hint: A quark or lepton is called *active at a scale* μ as long as it satisfies $m \le \mu$. Note that to very good approximation the matching condition $\alpha_i^{N_f=N}(\mu = m) = \alpha_i^{N_f=N+1}$ holds. In this exercise you will have to think about the number of active quarks and leptons on your own, depending on the scale μ.

a) Show that Eq. (3.1) implies

$$\alpha_i(\mu) = \frac{\alpha_i(\mu_0)}{1 + \frac{\alpha_i(\mu_0)}{4\pi}\, \beta_0^i \ln\left(\frac{\mu^2}{\mu_0^2}\right)}\,. \tag{3.4}$$

Look up the measured value $\alpha_s(M_Z)$ at the Z boson mass M_Z in the PDG.

b) Sketch the scale dependence of α_s for scales $\mu < M_Z$. Use a fixed number of active quarks, e.g. $N_f = 5$, for simplicity. At which scale $\mu = \Lambda$ does $\alpha_s(\mu)$ diverge?

c) At which scale $\mu = \mu_V$ is $\alpha_s(\mu_V) = \alpha_e(\mu_V)$?