9. Übungsblatt zur Vorlesung — WS 2019/20

Einführung in die Elementarteilchentheorie — **Prof. G. Hiller**

Abgabe: bis Di. 10. Dezember 2019, 12:00 Uhr — **Kasten 245**

http://people.het.physik.tu-dortmund.de/~ghiller/WS1920ETT.html

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Exercise 1: Adjoint representation of $SU(N)$ **(8 Points)**

The generators T^a of the group $SU(N)$ are hermitian operators which construct all infinitesimal group transformations. The commutator of two generators can be written as a linear combination

$$\left[T^a, T^b\right] = \mathrm{i}\, f^{abc}\, T^c\,, \tag{1.1}$$

where f^{abc} are structure constants. The commutator relation (1.1) and the vector space spanned by the generators are called the *Lie algebra* of the corresponding Lie group $SU(N)$.

(a) Derive the *Jacobi identity*

$$\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] = 0 \tag{1.2}$$

using the following relation

$$f^{ade}\, f^{bcd} + f^{bde}\, f^{cad} + f^{cde}\, f^{abd} = 0\,. \tag{1.3}$$

(b) For every simple Lie algebra an adjoint representation can be defined. The corresponding generators can be written as

$$\left(t^b\right)_{ac} = \mathrm{i}\, f^{abc}\,. \tag{1.4}$$

Show that these generators satisfy the Lie algebra

$$\left(\left[t^a, t^c\right]\right)_{be} = \mathrm{i}\, f^{acd} \left(t^d\right)_{be}\,. \tag{1.5}$$

Use Eq. (1.3) and keep in mind that the structure constants are anti-symmetric in the first two indices, i.e. $f^{abc} = -f^{bac}$.

(c) The quadratic Casimir invariant of a representation R is defined as

$$C(R)\mathbb{1} = \sum_a \left(t^a\, t^a\right)\,, \tag{1.6}$$

where t^A denotes the generators of the gauge group in a representation R. The Dynkin index of a representation R reads

$$S(R)\,\delta^{ab} = \mathrm{Tr}\left(t^a\, t^b\right)\,. \tag{1.7}$$

Show that for the adjoint representation of the $SU(3)$ the following relation holds

$$C(\mathrm{adj}) = S\left(\mathrm{adj}\right) = 3\,, \tag{1.8}$$

taking into consideration the non-vanishing $SU(3)$ structure constants

$$f^{123} = 1\,,\, f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}\,,\, f^{458} = f^{678} = \frac{\sqrt{3}}{2}\,. \tag{1.9}$$

Exercise 2: Parton distribution functions **(6 Points)**

a) On the website `http://hepdata.cedar.ac.uk/pdf/pdf3.html` you can find the parton distribution functions $xf(x,Q^2)$ as a function of the Bjorken variable x. Use the data of the (G)JR group, set JR09FFnnlo, and plot the parton distribution functions of the quarks and gluons with $Q^2 \in \{0.1, 1, 10\}$ GeV^2. Interpret the plots.

b) Sum rules can be derived for the parton distribution functions. From the definition of the total momentum of the proton one finds

$$\int_0^1 x \sum_i f_i(x) = 1\,. \tag{2.1}$$

Consider the distribution functions $u_v(x)$, $d_v(x)$, $s(x)$, $\bar{s}(x)$ and $g(x)$, where the index v denotes the valence quarks and non-indexed quarks are virtual sea quarks. What are the sum rules of the electric charge $Q = +1$ and the vanishing strangeness $S = 0$ of the proton?

Exercise 3: Isospin rotations for quarks and antiquarks **(6 Points)**

The up- and down-quark fields can be written as SU(2)-Isospin doublets $q = (u,d)^T$, where for a general isospin doublet $(q_1,q_2)^T$ the components q_1 and q_2 fulfil $I_3(q_1) = 1/2$ and $I_3(q_2) = -1/2$. Here I_3 denotes the third component of the isospin. In the following we want to investigate the transformation of such a doublet under a rotation U in isospin space. The rotation is given by

$$q' = Uq \quad \text{with} \quad U = \exp\left(i\vec{\phi}\vec{\tau}\right), \tag{3.1}$$

with angles ϕ_i and rotation generators $\tau_i = \sigma_i/2$, where σ_i are the Pauli matrices.

a) Calculate U and $q' = Uq$ for a rotation around $\phi = (0,\pi,0)^T$.

b) Consider now the antiquark doublet. The antiquarks belong to the so-called conjugate representation of SU(2), so that for the isospin part of the fields the relations "$\bar{u} = u^*$" and "$\bar{d} = d^*$" apply (the quotation marks indicate that the relations are not valid for the entire fields, but only for their respective isospin part). Because of the Gell-Mann-Nishijima relation, $I_3(\bar{u}) = -I_3(u)$ and $I_3(\bar{d}) = -I_3(d)$ hold as well. Therefore, the ansatz $q = (a\bar{d}, b\bar{u})^T$ may be used for complex numbers a, b with $|a| = |b| = 1$. Proceed as follows: Conjugate equation (3.1) and express the result using this ansatz for the antiquark doublets. How should you choose a and b so that the antiquark doublet is actually a doublet, that is, $\bar{q}' = U\bar{q}$ is fulfilled?

Note: Use your result from subtask a) to determine the coefficients.