6. Übungsblatt zur Vorlesung — WS 2019/20

Einführung in die Elementarteilchentheorie — **Prof. G. Hiller**

Abgabe: bis Di. 19. November 2019, 12:00 Uhr — **Kasten 245**

http://people.het.physik.tu-dortmund.de/~ghiller/WS1920ETT.html

---

Exercise 1: Extended QED: Everything that's (not) allowed **(7 Points)**

The QED Lagrangian, which describes photons, charged spin-1/2 fermions and their interaction via electromagnetism, is given by

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(\mathrm{i}\not{D} - m)\,\psi\,, \tag{1.1}$$

where $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$ is the covariant derivative. We now try to add additional terms to the Lorentz-invariant QED Lagrangian

$$\mathscr{L}_{\text{QED}} \to \mathscr{L}_{\text{QED}} + \mathscr{L}_1\,, \tag{1.2}$$

$$\mathscr{L}_1 = -\frac{1}{4} F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + \frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 + \bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} + \bar{\psi}\psi F_{\mu\nu}F^{\mu\nu} - A_\mu F^{\mu\nu} A_\nu\,, \tag{1.3}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and m, λ are constants.

(a) The extended Lagrangian in Eq. (1.2) needs to fulfill $U(1)$-gauge invariance. Which terms in Eq. (1.3) violate this invariance? Such terms must not appear in the fundamental Lagrangian!

(b) The most powerful tool to restrict new or additional terms is *renormalization*, which forbids all terms with mass dimension $d > 4$ in a quantum field theory in $3+1$ Minkowski space.

Employ Eq. (1.1) to calculate the canonical mass dimensions of the fields ψ, A_μ and identify the non-renormalizable terms in Eq. (1.3).

Hint: The action $S = \int \mathrm{d}^4x \mathscr{L}(x)$ has mass dimension $[S] = 0$. As the mass dimension of a length is inverse proportional to a mass, we can infer $\left[\mathrm{d}^4x\right] = -4$ as well as $[\mathscr{L}(x)] = +4$ and $\left[D_\mu\right] = +1$.

(c) The first term in Eq. (1.3) fulfils every requirement from (a) and (b). However, is must not appear in Eq. (1.1)! Why is that? Which exact symmetry (in QED) is broken in that case?

(d) Find an expression K^μ which obeys the following relation:

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu\,. \tag{1.4}$$

Why can one neglect such a term $F_{\mu\nu}\tilde{F}^{\mu\nu}$ in the Lagrangian?

(e) Why does the last term in Eq. (1.3), $A_\mu F^{\mu\nu} A_\nu$, vanish?

Exercise 2: Feynman rules **(7 Points)**

Consider the following Lagrangian density involving a real scalar field Φ and a Dirac field ψ:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - i\,g\,(\bar{\psi}\gamma_5\psi)\,\Phi\,, \tag{2.1}$$

where M and m are the masses of Φ and ψ, respectively, and g is a coupling constant.

(a) Show the following contractions:

$$a_{\vec{p},\sigma}\overline{\psi}(x) \equiv \{a_{\vec{p},\sigma}, \overline{\psi}(x)\} = \frac{1}{\sqrt{2E_p}}\overline{u}_\sigma(\vec{p})e^{ipx}\,, \tag{2.2}$$

$$b_{\vec{p},\sigma}\psi(x) \equiv \{b_{\vec{p},\sigma}, \psi(x)\} = \frac{1}{\sqrt{2E_p}}v_\sigma(\vec{p})e^{ipx}\,, \tag{2.3}$$

$$\Phi(x)a^\dagger_{\Phi,\vec{p}} \equiv \left[\Phi(x), a^\dagger_{\Phi,\vec{p}}\right] = \frac{1}{\sqrt{2E_p}}e^{-ipx}\,. \tag{2.4}$$

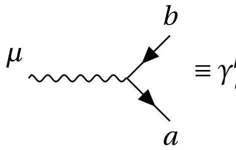
(b) Determine the Feynman rule in momentum space for the vertex $\bar{\psi}\psi\Phi$. Consider the decay of $\Phi(\vec{p}_1) \to \psi(\vec{p}_2)\bar{\psi}(\vec{p}_2)$ and use the Wick theorem to compute

$$< \vec{p}_2\vec{p}_3|S - I|\vec{p}_1 >= i\int d^4x < \vec{p}_2\vec{p}_3|N[-i\,g\,(\bar{\psi}\gamma_5\psi)\,\Phi]|\vec{p}_1 >\,, \tag{2.5}$$

where $N[...]$ is the normal ordered product, defined in Ex. 1 on the last sheet.

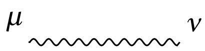
Exercise 3: Feynman diagrams playground **(6 Points)**

In this exercise we introduce diagrammatic notations similar to the famous *Feynman rules*. The individual parts of the diagram are depicted and described as follows.

vertex: 

$\equiv \gamma^\mu_{ab}$ (3.1)

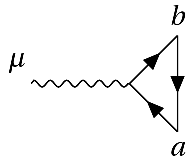
propagators: (3.2)



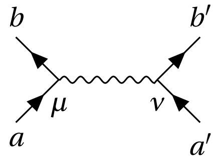
$\equiv g_{\mu\nu}$

$\equiv \mathbb{1}_{ab}$

(Note that these Feynman rules are constructed as an exercise and do *not* represent the true Feynman rules.) Here, a, b and μ, ν denote Spinor and Lorentz indices, respectively. A full diagram is constructed by these individual parts as follows. The parts are *glued together* by summing over the indices of the interfaces (*vertices*). Vertices can only be connected to propagators and vice versa. Thus, we can construct diagrams, which then can be translated into mathematical expressions. As an example, consider the following formal diagrams which can be translated as



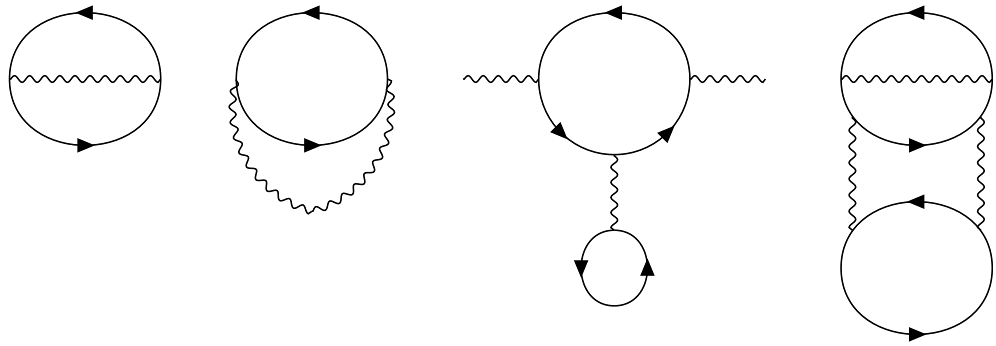
$$\equiv \sum_{a,b} \mathbb{1}_{ab}\,\gamma^{\mu}_{ba} = \mathrm{Tr}\left(\gamma^{\mu}\right)\,, \tag{3.3}$$



$$\equiv \sum_{\mu,\nu} \gamma^{\mu}_{ba}\,\gamma^{\nu}_{b'a'}\,g_{\mu\nu} = \gamma^{\mu}_{ba}\gamma_{\mu,b'a'}\,. \tag{3.4}$$

(3.5)

Here, we follow lines in the opposite direction to the arrows.
Use this notation and compute the following diagrams:



(3.6)