5. Übungsblatt zur Vorlesung WS 2019/20

Einführung in die Elementarteilchentheorie **Prof. G. Hiller**
Abgabe: bis Di. 12. November 2019, 12:00 Uhr **Kasten 245**
http://people.het.physik.tu-dortmund.de/~ghiller/WS1920ETT.html

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Exercise 1: Wick's theorem **(5 Points)**

Prove Wick's theorem for two *fermionic* fields, i.e. show that

$$T\{\psi(x)\overline{\psi}(y)\} = N\{\psi(x)\overline{\psi}(y)\} + \contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y)\,, \tag{1.1}$$

with the contraction of two fermionic fields

$$\contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y) = \begin{cases} \{\psi^+(x), \overline{\psi}^-(y)\} & \text{for } x^0 > y^0 \\ -\{\overline{\psi}^+(y), \psi^-(x)\} & \text{for } y^0 > x^0 \end{cases} . \tag{1.2}$$

Hints:
For two *scalar* fields the corresponding relation between normal-ordered and time-ordered product reads

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y)\} + \contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y)\,, \tag{1.3}$$

where $\phi(x) = \phi^+(x) + \phi^-(x)$ is the contraction defined as

$$\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases} . \tag{1.4}$$

In the normal-ordered product $N\{...\}$ all annihilation operators are moved to the right of all creation operators, e.g.

$$N\{a_k a_p^\dagger a_q\} = a_p^\dagger a_k a_q\,. \tag{1.5}$$

In order to formulate Wick's theorem for *fermionic* fields, the time-ordered and normal-ordered product need to be generalized for *fermions*. The time-ordered product obtains a minus sign for each commutation of operators

$$T\{\psi(x)\overline{\psi}(y)\} = \begin{cases} \psi(x)\overline{\psi}(y) & \text{for } x^0 > y^0 \\ -\overline{\psi}(y)\psi(x) & \text{for } y^0 > x^0 \end{cases} . \tag{1.6}$$

The same is true for the normal-ordered product. For example

$$N\{a_k a_q a_p^\dagger\} = (-1)^2 a_p^\dagger a_k a_q. \tag{1.7}$$

Because $\psi^+ \mid 0\rangle = 0$ and $\langle 0 \mid \psi^- = 0$, it is appropriate to separate into positive and negative frequencies:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s [\underbrace{a_{p,s} u_s(p) e^{-ipx}}_{\propto \psi^+(x)} + \underbrace{b_{p,s}^\dagger v_s(p) e^{ipx}}_{\propto \psi^-(x)}], \tag{1.8}$$

$$\overline{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s [\underbrace{b_{p,s} \overline{v}_s(p) e^{-ipx}}_{\propto \overline{\psi}^+(x)} + \underbrace{a_{p,s}^\dagger \overline{u}_s(p) e^{ipx}}_{\propto \overline{\psi}^-(x)}]. \tag{1.9}$$

Exercise 2: Helicity of the photon **(5 Points)**

The helicity of a photon H is defined as the projection of the angular momentum vector $\vec{J}$ of the photon on the direction of its momentum $\vec{k}$:

$$H = \vec{J} \cdot \frac{\vec{k}}{|\vec{k}|} \,. \tag{2.1}$$

Construct the polarization vectors ϵ^μ of the photon so that

$$H\epsilon^\mu = h\epsilon^\mu \,, \quad \epsilon^*_\mu \epsilon^\mu = -1 \,. \tag{2.2}$$

The angular momentum operator for four vectors is given by

$$\vec{J} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \vec{\tilde{J}} & \\ 0 & & & \end{pmatrix} \,, \tag{2.3}$$

with

$$\vec{\tilde{J}} = (\tilde{J}_1, \tilde{J}_2, \tilde{J}_3) \,, \tag{2.4}$$

The generators of rotations in the adjoint representation are defined by ($a, b, c \in \{1, 2, 3\}$)

$$(\tilde{J}_a)_{bc} = -\mathrm{i}\epsilon_{abc} \,. \tag{2.5}$$

The polarization vectors are eigenvectors of the helicity operator H of a photon, which is moving in z-direction with four momentum $k^\mu = (E, 0, 0, E)$.

Exercise 3: Proca Lagrangian **(5 Points)**

A theory with a massive vector field is described by the Proca Lagrangian

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2 A^\mu A_\mu - A^\mu j_\mu \,. \tag{3.1}$$

Prove that the corresponding propagator in momentum space is given by

$$D_{\mu\nu}(k) = \frac{i}{k^2 - M^2}\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right) \tag{3.2}$$

To do so perform the following steps:

(a) Calculate the equations of motion (EOM) for the vector field A_μ.

(b) Compute the propagator by replacing the inhomogeneous part of the EOM by $\mathrm{i}g^{\mu\nu}\delta^{(4)}(x)$. Besides the factor i, the propagator is the Greens function of the EOM. You can solve this equation by Fourier transforming and using the ansatz

$$D_{\mu\nu}(k) = ak_\mu k_\nu + bg_{\mu\nu} \,, \tag{3.3}$$

where a and b are constants you need to find. This ansatz is the most general form of a tensor of rank two, that only depends on k.

Exercise 4: QED Lagrangian **(5 Points)**

The QED Lagrangian $\mathscr{L}_{\text{QED}}$ reads

$$\mathscr{L}_{\text{QED}} = \overline{\psi}(x)(\mathrm{i}\partial_\mu\gamma^\mu - m)\psi(x) - e\overline{\psi}(x)A^\mu\gamma_\mu\psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{4.1}$$

where ψ is the fermion filed, $\bar{\psi} = \psi^\dagger\gamma^0$, A_μ is the covariant four-potential of the electromagnetic field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strenght tensor.

(a) Show that $\mathscr{L}_{\text{QED}}$ is invariant under a local $U(1)$ gauge transformation

$$\psi(x) \to \exp(i\alpha(x))\psi(x)\,, \tag{4.2}$$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)\,. \tag{4.3}$$

(b) On the last sheet you have shown that the transformation

$$\mathscr{L}(\phi) \to \mathscr{L}'(\phi) = \mathscr{L}(\phi) + \partial^\mu f_\mu(\phi(x)) \tag{4.4}$$

with an arbitrary four-current f_μ does not change the physics of $\phi(x)$, where ϕ is an arbitrary field. Find a function $f_\mu(\psi,\bar{\psi})$ that symmetrizes the Lagrangian in Eq. (4.1) in ψ and $\bar{\psi}$, i.e.

$$\begin{aligned}\mathscr{L}_{\text{QED}} \to \mathscr{L}'_{\text{QED}} = &\frac{1}{2}\overline{\psi}(x)\mathrm{i}\partial_\mu\gamma^\mu\psi(x) - \frac{1}{2}\left(\partial_\mu\overline{\psi}(x)\right)\mathrm{i}\gamma^\mu\psi(x) - m\overline{\psi}(x)\psi(x)\\ &- e\overline{\psi}(x)A^\mu\gamma_\mu\psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,.\end{aligned} \tag{4.5}$$

Calculate the equations of motion (EOM) for ψ, $\bar{\psi}$ and A_μ. Why can ψ and $\bar{\psi}$ be varied independently?

(c) The Noether theorem implies a conservation law for any differentiable symmetry of the action. Show that $U(1)$ invariance implies charge conservation. To do so, first show that the Noether current $j^\mu = e\overline{\psi}(x)\gamma^\mu\psi(x)$ is conserved.