4. Übungsblatt zur Vorlesung — WS 2019/20

Einführung in die Elementarteilchentheorie — **Prof. G. Hiller**

Abgabe: bis Di. 05. November 2019, 12:00 Uhr — **Kasten 245**

http://people.het.physik.tu-dortmund.de/~ghiller/WS1920ETT.html

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Exercise 1: Helicity **(5 Points)**

The helicity operator for fermions with three-momentum $\vec{p}$ is given by

$$h = \frac{1}{2}\frac{\vec{p}}{|\vec{p}|}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}. \tag{1.1}$$

(a) What happens when the operator h acts on the spinors $u_i(p)$ and $v_i(p)$ (defined on sheet 3)?

(b) Consider the relativistic limit $m/E \to 0$ and repeat the computation from part (a). Discuss your results.

Exercise 2: How NOT to quantize a Dirac field **(10 Points)**

Why do we use ANTI-commutation relations for fermions

$$\left\{\psi_a(x), \psi_b^\dagger(y)\right\} = \delta^{(3)}\left(\vec{x}-\vec{y}\right)\delta_{ab}, \tag{2.1}$$

at equal times $t = x_0 = y_0$, with spinor components a and b?

(a) First quantize the scalar field, starting with the Lagrange density of the Klein-Gordon-Field

$$\mathscr{L}_{KG} = \frac{1}{2}\left[\left(\partial_\mu\phi(x)\right)\left(\partial^\mu\phi(x)\right) - m^2\phi(x)^2\right]. \tag{2.2}$$

To do so, calculate the conjugated field $\pi(x)$ corresponding to

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ipx} + a_p^\dagger e^{+ipx}\right), \tag{2.3}$$

and prove the commutation relation $[\phi(x), \pi(x)] = i\delta^{(3)}\left(\vec{x}-\vec{y}\right)$ at equal times $t = x_0 = y_0$ by using the commutation relations $\left[a_p, a_q^\dagger\right] = (2\pi)^3\delta^{(3)}\left(\vec{p}-\vec{q}\right)$.

(b) Now suppose, analogous to (a), that fermionic states are symmetric, e.g. calculate the commutator

$$[\psi_a(x), \psi_b^\dagger(y)] \quad \text{with} \quad t = x_0 = y_0. \tag{2.4}$$

Use the Fourier decompositions

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s [\underbrace{a_{p,s}u_s(p)e^{-ipx}}_{\propto\,\psi^+(x)} + \underbrace{b_{p,s}^\dagger v_s(p)e^{ipx}}_{\propto\,\psi^-(x)}], \tag{2.5}$$

$$\overline{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s [\underbrace{b_{p,s}\overline{v}_s(p)e^{-ipx}}_{\propto\,\overline{\psi}^+(x)} + \underbrace{a_{p,s}^\dagger\overline{u}_s(p)e^{ipx}}_{\propto\,\overline{\psi}^-(x)}], \tag{2.6}$$

as well as

$$\left[a_{p,r}, a^\dagger_{q,s}\right] = \left[b_{p,r}, b^\dagger_{q,s}\right] = (2\pi)^3 \delta^{(3)}\left(\vec{p} - \vec{q}\right)\delta_{rs}. \tag{2.7}$$

The indices p, q denote the momenta of the fermions and r, s their spin states. Compare your results to the relation in Eq. (2.1). Show that Eq. (2.5)) and (2.6) can not be used as the Fourier decompositions in this case.

(c) Now employ

$$\left[a_{p,r}, a^\dagger_{q,s}\right] = \left[b_{p,r}, b^\dagger_{q,s}\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\delta_{rs}, \tag{2.8}$$

and the following associated Fourier decompositions of the fields ψ and $\overline{\psi}$

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_{p,s} u_s(p) e^{-ipx} + b_{p,s} v_s(p) e^{ipx} \right), \tag{2.9}$$

$$\overline{\psi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a^\dagger_{p,s} \overline{u}_s(p) e^{ipx} + b^\dagger_{p,s} \overline{v}_s(p) e^{-ipx} \right). \tag{2.10}$$

What do the expressions

$$\langle 0|\psi(x)\overline{\psi}(y)|0\rangle \quad \text{and} \quad \langle 0|\overline{\psi}(y)\psi(x)|0\rangle \tag{2.11}$$

imply in this case? Which problem do you encounter concerning causality? You do not need to perform a complete calculation for this task.

(d) Finally use Eq. (2.9) and (2.10) to calculate the Hamilton function

$$H = \int d^3x\, \mathscr{H}, \tag{2.12}$$

using the Lagrangian density of a free Dirac field

$$\mathscr{L}_D = \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x). \tag{2.13}$$

Which problem do you encounter?

(e) The problem is solved by assuming

$$\left\{a_{p,r}, a^\dagger_{q,s}\right\} = \left\{b_{p,r}, b^\dagger_{q,s}\right\} = (2\pi)^3 \delta^{(3)}\left(\vec{p} - \vec{q}\right)\delta_{rs}. \tag{2.14}$$

How do the Fourier decompositions of ψ and $\overline{\psi}$ look like? Show that this ansatz gives the correct commutation relations and solves the problem encountered in part (d).

Exercise 3: Lagrange densities **(5 Points)**

(a) Derive the Euler-Lagrange equations for a Lagrangian density $\mathscr{L}\left[\phi(x), \partial_\mu\phi(x)\right]$ using the principle of stationary action:

$$0 = \delta \int \mathrm{d}^4x\, \mathscr{L}\left[\phi(x), \partial_\mu\phi(x)\right], \tag{3.1}$$

and show that the transformation

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} + \partial^\mu f_\mu(\phi(x)) \tag{3.2}$$

with an arbitrary four-current f_μ does not change the physics of $\phi(x)$.
Hint: You can use the following generalization of the Gaussian theorem in R^3 to Minkowski space

$$\int_G \mathrm{d}^4 x \partial_\mu f(\phi(x), \partial_\mu \phi(x)) = \int_{\partial G} \mathrm{d}\sigma_\mu f(\phi(x), \partial_\mu \phi(x)), \tag{3.3}$$

where G is the volume of integration in Minkowski space and σ_μ the normal on the surface ∂G.

(b) The Lagrangian of the free Dirac field reads

$$\mathscr{L}_0 = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x). \tag{3.4}$$

Show that the Euler-Lagrange equations of motion are equivalent to the Dirac equation.

(c) The Noether theorem implies a conservation law for any differentiable symmetry of the action. Show explicitly that the Lagrangian in Eq. (3.4) is invariant (i.e. $\delta\mathscr{L} = 0$) under an infinitesimal transformation

$$\psi \to \psi + \delta\psi, \quad \text{with} \quad \delta\psi = i\epsilon e\psi, \tag{3.5}$$

where ϵ is an infinitesimal parameter and e is an arbitrary real parameter. Compute the Noether current

$$j^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)} \delta\psi \tag{3.6}$$

and show that it is conserved if the fields fulfill the equations of motion.