3. Übungsblatt zur Vorlesung — WS 2019/20

Einführung in die Elementarteilchentheorie — **Prof. G. Hiller**

Abgabe: bis Di. 29. Oktober 2019, 12:00 Uhr — **Kasten 245**

http://people.het.physik.tu-dortmund.de/~ghiller/WS1920ETT.html

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Exercise 1: Dirac algebra **(8 Points)**

The *Dirac equation* is a relativistic equation of motion for fermions,

$$\left(\not{p} - m\right)\psi(p) = 0\,, \tag{1}$$

where $\not{p} = p^\mu \gamma_\mu$ and the gamma matrices γ^μ satisfy the anticommutator relation

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}\,. \tag{2}$$

In the following, we give a list of tensors which form a complete basis and can be constructed out of gamma matrices:

tensor	degrees of freedom
$\mathbb{1}$	1
γ_μ	4
$\gamma_{[\mu}\gamma_{\nu]} = -\mathrm{i}\sigma_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\omega}\sigma^{\rho\omega}\gamma^5$	6
$\gamma_{[\mu}\gamma_\nu\gamma_{\rho]} = -\mathrm{i}\varepsilon_{\mu\nu\rho\omega}\gamma^\omega\gamma^5$	4
$\gamma_{[\mu}\gamma_\nu\gamma_\rho\gamma_{\omega]} = -\mathrm{i}\varepsilon_{\mu\nu\rho\omega}\gamma^5$	1

(3)

with $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$. We define the antisymmetric product as

$$A_{[a_1,\dots,a_n]} \equiv \frac{1}{n!}\left(\sum_{\text{even perm. in } a_i} A_{a_1,\dots,a_n} - \sum_{\text{odd perm. in } a_i} A_{a_1,\dots,a_n}\right). \tag{4}$$

Since the tensors in Eq. (3) form a basis, every product M of gamma matrices can be written as a linear combination of those 16 tensors:

$$M = \sum_a M^a \Gamma_a\,, \tag{5}$$

where $\Gamma_a = \{\mathbb{1},\, \gamma_\mu,\, \gamma_{[\mu}\gamma_{\nu]},\, \gamma_{[\mu}\gamma_\nu\gamma_{\rho]},\, \gamma_{[\mu}\gamma_\nu\gamma_\rho\gamma_{\omega]}\}$ and M^a denote the expansion coefficients. While all tensors can be separated into a symmetric and an antisymmetric part, it can be done more efficiently and easily in this particular basis.
The following relations hold:

$$\mathrm{Tr}\left(\gamma^\mu\right) = 0\,, \quad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}\,, \quad \{\gamma^\mu, \gamma^5\} = 0\,. \tag{6}$$

a) Show that

$$\gamma^5 = -\frac{\mathrm{i}}{4!}\varepsilon_{\mu\nu\rho\omega}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\omega\,. \tag{7}$$

b) Use Eqs. (6) and (7) to show that

$$\gamma_{[\mu}\gamma_\nu\gamma_{\rho]} = -\mathrm{i}\varepsilon_{\mu\nu\rho\omega}\gamma^\omega\gamma^5\,. \tag{8}$$

c) An alternative basis is $\Gamma_a = \{\mathbb{1},\ \gamma_\mu,\ \sigma_{\mu\nu},\ \mathrm{i}\gamma_\mu\gamma^5,\ \gamma^5\}$. Just as the basis in Eq. (3), these matrices form an orthonormal basis with respect to the scalar product

$$\langle \Gamma_a, \Gamma_b \rangle \equiv \frac{1}{4} \operatorname{Tr}\left(\Gamma_a \Gamma^b\right). \tag{9}$$

Prove that the aforementioned basis is indeed orthonormal, i.e. show that

$$\langle \Gamma_a, \Gamma_b \rangle = \delta_{ab}. \tag{10}$$

Hints:

$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -4!\,, \tag{11}$$

$$\varepsilon^{\alpha\beta\gamma\mu}\varepsilon_{\alpha\beta\gamma\nu} = -6\delta^\mu_\nu\,, \tag{12}$$

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right), \tag{13}$$

$$\varepsilon^{0123} = -\varepsilon_{0123} = +1\,. \tag{14}$$

In general, the following relation holds in n-dimensional Minkowski space:

$$\varepsilon^{\mu_1\ldots\mu_m\alpha_{m+1}\ldots\alpha_n}\varepsilon_{\nu_1\ldots\nu_m\alpha_{m+1}\ldots\alpha_n} = -m!(n-m)!\,\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_m]}_{\nu_m}\,. \tag{15}$$

Exercise 2: Lorentz transformations of spinors **(6 Points)**

Left and right-handed *Weyl spinors* $\phi_{L/R}$ are two-component mathematical objects which are defined by their Lorentz transformation properties. They transform under a proper, orthochronous Lorentz transformation $\Lambda \in L_+^\uparrow$ as

$$\phi_{L/R}(x) \to S_{L/R}(\Lambda)\,\phi_{L/R}\left(\Lambda^{-1}x\right). \tag{16}$$

The transformations $S_{L/R}(\Lambda)$ are functions of rotation angles $\vec{\theta}$ and rapidities $\vec{\eta}$, which parametrize the Lorentz transformations $\Lambda = \Lambda\left(\vec{\theta}, \vec{\eta}\right)$. We can write

$$S_{L/R}(\Lambda) = \exp\left(-\mathrm{i}\frac{\vec{\theta}\cdot\vec{\sigma}}{2} \mp \frac{\vec{\eta}\cdot\vec{\sigma}}{2}\right), \tag{17}$$

where σ^i denote the Pauli matrices which satisfy the anticommutator relation

$$\left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij}\,. \tag{18}$$

Additional information/Context:
Dirac spinors ψ are four-component objects and are composed by one left and one right-handed Weyl spinor:

$$\psi = \begin{pmatrix}\phi_L \\ \phi_R\end{pmatrix}. \tag{19}$$

The dependence of left and right-handed components of a particle are determined by the Dirac equation. Then, left and right-handed states can be obtained by applying the projection operator

$$P_{L/R} = \frac{1}{2}\left(1 \mp \gamma^5\right) \tag{20}$$

onto the Dirac spinors.

a) Show that the following equation holds for arbitrary $\vec{\theta}$ and $\vec{\eta}$:

$$\det S_{L/R}(\Lambda) = 1\,. \tag{21}$$

b) Calculate $S_{L/R}(\Lambda)$ in the case of a rotation along the z-axis, i.e. $\vec{\theta} = \theta_z \vec{e}_z$ and $\vec{\eta} = \vec{0}$. Furthermore, compute $S_{L/R}(\Lambda)$ for $\theta_z = 2\pi$ and $\theta_z = 4\pi$. Compare your results to those of an analogous rotation of a vector in $\mathbb{R}^3$. What do you notice?

c) Show that the following equation holds for Lorentz boosts with $\vec{\theta} = 0$:

$$S_{L/R}(\Lambda) = \exp\left(\mp \frac{\vec{\eta}\cdot\vec{\sigma}}{2}\right) = \mathbb{1}\cosh\left(\frac{|\vec{\eta}|}{2}\right) \mp \frac{\vec{\sigma}\cdot\vec{\eta}}{|\vec{\eta}|}\sinh\left(\frac{|\vec{\eta}|}{2}\right). \tag{22}$$

Afterwards, calculate $S_{L/R}(\Lambda)$ for a Lorentz boost from the rest frame into a reference frame with momentum $\vec{p}$.

Exercise 3: Dirac spinors **(6 Points)**

Solutions of the Dirac equation are defined as $\psi_j^+(p) = u_j(p)\exp(-\mathrm{i}px)$ and $\psi_j^-(p) = v_j(p)\exp(\mathrm{i}px)$, with $j = 1, 2$ and Dirac spinors

$$u_j(p) = \frac{\not{p} + m}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_j\\ \xi_j\end{pmatrix}, \quad v_j(p) = \frac{-\not{p} + m}{\sqrt{2(E+m)}}\begin{pmatrix}\xi_j\\ -\xi_j\end{pmatrix}, \tag{23}$$

$$\xi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_2 = \begin{pmatrix}0\\1\end{pmatrix}. \tag{24}$$

Show the following relations for spinors with four-momentum p, where $p^2 = m^2$ and $p^0 = E > 0$:

a) $\bar{u}_i(p)u_j(p) = 2m\delta_{ij}, \quad \bar{v}_i(p)v_j(p) = -2m\delta_{ij}$ (normalization),

b) $\bar{u}_i(p)v_j(p) = \bar{v}_i(p)u_j(p) = 0$ (orthogonality),

c) $\sum_i u_i(p)\bar{u}_i(p) = \not{p} + m, \quad \sum_i v_i(p)\bar{v}_i(p) = \not{p} - m$ (completeness),

where $\bar{u} = u^\dagger\gamma^0$, $\not{p} = p^\mu\gamma_\mu$ and

$$\gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}, \quad \left(\sigma^i\right)^2 = \mathbb{1}\,. \tag{25}$$